On minimal quasi-ideals and minimal bi-ideals in compact semigroups

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The concept of quasi-ideals and bi-ideals in semigroups has been introduced respectively by O. STEINFELD in [6] and R. A. GOOD and D. R. HUGHES in [2]. Both notions have been generalized by S. LAJOS in (4) to the so-called (m, n)-quasi-ideals and (m, n)-bi-ideals in a semigroup. Besides other interesting properties, this author has proved that if S is a regular semigroup, then each (m, n)-bi-ideal is an (m, n)-quasi-ideal. In [3] K. M. KAPP has shown that if any element of a bi-ideal B in a semigroup S is regular, then B is a quasi-ideal.

In this paper we prove that if S is a compact semigroup, then it contains as well minimal quasi-ideals as minimal bi-ideals and it is moreover shown that the sets of minimal quasi-ideals and minimal bi-ideals coincide.

Let us recall the definitions of a quasi-ideal and a bi-ideal in a semigroup S.

Definition 1. Let S be a semigroup; then

(i) a non empty subset Q of S is a *quasi-ideal* of S if $QS \cap SQ \subset Q$,

(ii) a non empty subset B of S is a *bi-ideal* of S if $B^2 \cup BSB \subset B$,

(iii) a quasi-ideal (bi-ideal) in a semigroup S is called *minimal* if it does not properly contain any quasi-ideal (bi-ideal) of S.

In the sequel we admit that S is a compact semigroup (also called a *compact mob*) which means that

α) S is a compact Hausdorff space,

β) $(x, y) \to x \cdot y$ is continuous on $S \times S$ (see also [5], p. 17).

We now first establish two theorems concerning the existence of minimal quasi-ideals and minimal bi-ideals in a compact mob, the proof of which runs in the same way. By this reason we prove the first of them only.

Theorem 1. *Let S be a compact mob and let Q be a quasi-ideal in S; then Q contains a minimal quasi-ideal. Moreover each minimal quasi-ideal of S is closed.*

Proof. Call $\mathscr{U}$ the set of all closed quasi-ideals contained in Q; then $\mathscr{U}$ is non empty.

Indeed, if $x \in Q$, then $xS \cap Sx$ is a quasi-ideal contained in Q and since both xS and Sx are compact and hence closed, $xS \cap Sx$ is also closed. Let now $\mathscr{U}$ be partially ordered by inclusion and $(Q_i)_{i \in I}$ be a linearly ordered subcollection of $\mathscr{U}$; then $(Q_i)_{i \in I}$ is bounded below since, as S is compact, $\bigcap_{i \in I} Q_i$ is a non empty closed quasi-ideal contained in Q.

By means of Zorn's Lemma, $\mathscr{U}$ admits a minimal element, say Q_0 and we claim that Q_0 is a minimal quasi-ideal in S.

Indeed, assume that Q' is a quasi-ideal which is properly contained in Q_0; then for $x \in Q'$, $xS \cap Sx$ is a closed quasi-ideal in S and $xS \cap Sx \subset Q' \subset Q_0$, whence $Q' = Q_0 = Sx \cap xS$.

Theorem 2. *Let S be a compact mob and B be a bi-ideal of S; then B contains a minimal bi-ideal of S. Moreover each minimal bi-ideal of S is closed.*

Corollary 1. *Each minimal bi-ideal B of a compact mob is a quasi-ideal.*

Proof. Since B is a minimal bi-ideal of S, $B = aSa$ for all $a \in B$ and hence any element of B is regular. In view of [3] Prop. 1.9 it then follows that B is a quasi-ideal.

Corollary 2. *If B is a minimal bi-ideal of a compact mob S, then B is a (compact) topological group.*

Proof. Since B is a bi-ideal, for every $b \in B$, bB and Bb are bi-ideals contained in B. As B is minimal, $B = Bb = bB$ and so B is an abstract group. But B is also a compact mob so that, in virtue of [5], Th. 1.1.8, B is a topological group.

Theorem 3. *If $\mathscr{U}^*$ is the set of minimal quasi-ideals and $\mathscr{B}^*$ is the set of minimal bi-ideals of a compact mob S, then $\mathscr{U}^* = \mathscr{B}^*$.*

Proof. Let $B \in \mathscr{B}^*$; then by Corollary 1 of Theorem 2, B is a quasi-ideal of S and it hence contains a minimal quasi-ideal $Q \in \mathscr{U}^*$. But as each quasi-ideal is also a bi-ideal (see e.g. [1], Ex. 18 (a)), $B = Q \in \mathscr{U}^*$.

Conversely, let $Q \in \mathscr{U}^*$; then Q is a bi-ideal of S and it hence contains a minimal bi-ideal $B \in \mathscr{B}^*$. But again in view of Corollary 1, B is then a quasi-ideal contained in Q and so $Q = B \in \mathscr{B}^*$.

Corollary 1. *Let S be a commutative compact mob; then S contains only one minimal bi-ideal B which is also the only minimal quasi-ideal of S. Moreover $B = K$, the kernel of S.*

Proof. Since S is commutative, each quasi-ideal of S is an ideal of S and vice versa. Hence, as S contains only one minimal ideal, namely the kernel K of S (see e.g. [5], p. 32), $\mathscr{U}^* = \mathscr{B}^* = \{K\}$.

References

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