On groups and semigroups of spectral operators on a Banach space

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The results of this note complement some results of McCARTHY and STAMPFLI [4]. They proved that if $\{T(t): -\infty<t<\infty\}$ is a group of operators on a Hilbert space with $\|T(t)\|$ bounded on finite intervals, and if $T(t_0)$ is spectral (respectively scalar type) for some $t_0\neq 0$, then all the operators $T(t)$, for $-\infty<t<\infty$, are spectral (respectively scalar type).

In what follows X will be a complex Banach space. All operators are assumed to be bounded. We will denote the spectrum of an operator T by $\sigma(T)$, and its resolvent (evaluated at λ) by $R(\lambda; T)$. Our terminology concerning groups and semigroups of operators will be that of [1; Ch. VIII]. For definitions and results on spectral operators, we refer to [1; Ch. XV].

Theorem 1. *Let $\{T(t): t\geqq 0\}$ be a semigroup of operators with real spectra on a Banach space X such that $\|T(t)\|$ is bounded in finite intervals and $T(t_0)$ is one-to-one and scalar type for some $t_0\neq 0$. Then $T(t)$ is scalar type for every $t\geqq 0$, and the semigroup is strongly continuous.*

Proof. Without loss of generality we can take $t_0=1$, for otherwise we can consider the semigroup $[T(tt_0): t\geqq 0\}$.

First we prove the theorem in the case where $T(1)$ is invertible (in other words, we assume, for the present, that the semigroup can be extended to a group $\{T(t): -\infty<t<\infty\}$), let $E(\cdot)$ be the resolution of the identity of $T(1)$, and let $[a, b]\supset\sigma(T(1))$, where $b\geqq a>0$. Define $R(t)$ by

$$R(t) = \int \lambda^t \, dE(\lambda), \quad -\infty < t < \infty.$$

It is easy to verify that $\{R(t)\}$ is a group of operators with positive spectra which is uniformly bounded on finite intervals (as a matter of fact it is uniformly continuous). Each $T(s)$ commutes with $T(1)$, hence with $E(\cdot)$ and with every $R(t)$. If $U(t)=$

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$=R(-t)T(t)$, then $\{U(t)\colon -\infty<t<\infty\}$ is a periodic group of operators since $U(1)=I$. Also $\|U(t)\|$ is bounded in finite intervals, and hence is uniformly bounded by $M>0$. For any t, the spectral radius of $U(t)$ is $\leqq 1$ since $\{\|U^n(t)\|: n=1, 2, 3, \ldots\}$ is bounded. But the same is true for $(U(t))^{-1}=U(-t)$, therefore $\sigma(U(t))=\{1\}$ and $U(t)=I+N(t)$, where $N(t)$ is quasi-nilpotent. But since $U(t)$ is power bounded it follows from [2] that $N(t)=0$. Therefore $U(t)=I$, and

$$T(t) = \int \lambda^t dE(\lambda), \quad -\infty < t < \infty.$$

Now we prove the theorem in the general case. Let $\sigma(T(1))\subseteqq[0, b]$, and let $e_n=\left[\frac{1}{n}, b\right]$, $X_n=E(e_n)X$, and $X_0=\bigcup_{n=1}^{\infty} X_n$. For every t, $T(t)$ commutes with $E(e_n)$ and thus X_n is invariant under $T(t)$ and $R(\mu;\, T(t))$ for $\mu\in\varrho(T(t))$. If $T_n(t)=T(t)|X_n$ for $t\geqq 0$ then the semigroup $\{T_n(t)\}$ satisfies the hypothesis of the theorem and $T_n(1)$ is invertible and scalar type with resolution of the identity $E(\cdot)|X_n$. It follows, by the first part, that

$$T_n(t) = \int_{(1/n,\, b]} \lambda^t d(E(\lambda)|X_n).$$

Hence

$$T(t)x = \int_{(1/n,\, b]} \lambda^t dE(\lambda)x = \int_{(0,\, b]} \lambda^t dE(\lambda)x \text{ for } x\in X_n,$$

since

$$\int_{(0,\, 1/n)} \lambda^t dE(\lambda)x = \int_{(0,\, 1/n)} \lambda^t dE(\lambda)E(e_n)x = 0, \quad x\in X_n.$$

Therefore

$$T(t)x = \int_{(0,\, b]} \lambda^t dE(\lambda)x, \quad x\in X_0.$$

But X_0 is dense in X and $E(\{0\})=0$ since $T(1)$ is one-to-one. Hence, if $x\in X$, $E(e_n)x\to$ $\to E((0, b])x=x$. Therefore

$$T(t) = \int_{(0,\, b]} \lambda^t dE(\lambda).$$

It is now easy to show that $\{T(t)\}$ is strongly continuous.

Corollary. *If $\{T(t)\}$ is a semigroup of operators with real spectra on Hilbert space, and if $\|T(t)\|$ is bounded on finite intervals, $T(t_0)$ self-adjoint and one-to-one, for some $t_0\neq 0$, then every $T(t)$ is self-adjoint.*

Proof. If $E(\cdot)$ is the resolution of the identity for $T(t_0)$, then $E(\delta)$ is self-adjoint for every Borel set δ of the real line. From the proof of the theorem, $T(t)=$ $=\int \lambda^{t/t_0} dE(\lambda)$, and hence $T(t)$ is self-adjoint for $t\geqq 0$.

Theorem 2. *If $\{T(t)\colon t\geqq 0\}$ is a semigroup of operators with real spectra on X such that $\|T(t)\|$ is bounded on finite intervals and $T(t_0)$ is scalar type for some $t_0\neq 0$, then $T(t)$ is spectral for every $t\geqq 0$.*

Proof. Without loss of generality we can take $t_0=1$. Let $E(\cdot)$ be the resolution of the identity for $T(1)$ and $[0, b]\supseteq\sigma(T(1))$. Let $Z=E(\{0\})X$ and $Y=E((0, b])X$. Therefore $X=Y+Z$, and this sum is direct in both algebraic and topological senses; moreover both Y and Z are invariant under $T(t)$, for $t\geqq 0$, since $T(t)$ commutes with $E(\cdot)$. It is easy to see that $\{T(t)\,|\,Y\colon t\geqq 0\}$ is a semigroup satisfying the conditions of Theorem 1. Therefore

$$T(t)y = \int \lambda^t dE(\lambda)y, \quad y\in Y, \quad t\geqq 0.$$

Hence $T(t)E((0, b])$ is a scalar type operator.

On the other hand $\{T(t)|Z\colon t\geqq 0\}$ is a semigroup of operators on Z with $T(1)|Z=0$. Hence $T(t)|Z$ is nilpotent for $t>0$ since if $n>1/t$, then $(T(t)|Z)^n=0$. Therefore $T(t)E(\{0\})$ is nilpotent, for $t>0$. But $T(t)(=T(t)E((0, b])+T(t)E(\{0\}))$ is the sum of a scalar type operator and a nilpotent operator which commute with one another; hence it is spectral.

Theorem 3. *Let $\{T(t)\colon -\infty<t<\infty\}$ be a group of operators on X, having real spectra, with $\|T(t)\|$ bounded on finite intervals, and $T(1)$ spectral. Then every $T(t)$ is spectral $(-\infty<t<\infty)$.*

Proof. Let $E(\cdot)$ be the resolution of the identity for $T(1)$ and let N be its radical part. For every t, define $R(t)$ by $R(t)=(T(1))^t$. This is well-defined since the function $\lambda\to\lambda^t$ is analytic on a neighborhood of $\sigma(T(1))$. Moreover, $R(t)$ is a bounded spectral operator whose scalar part is $\int \lambda^t dE(\lambda)$, $\{R(t)\colon -\infty<t<\infty\}$ is a group of operators with real spectra, and $\|R(t)\|$ is bounded in finite intervals. For any real numbers s and t, $T(t)$ commutes with $T(1)$ and hence with $R(s)$. It follows that $\{R(-t)T(t)\colon -\infty<t<\infty\}$ is a group of operators, periodic, and uniformly bounded in norm. Therefore $T(t)=R(t)$, exactly as in the proof of Theorem 1. This proves the theorem.

The following two examples are taken from McCarthy and Stampfli [4] where they were used to show the sharpness of their results. They are given here too because they also show that our results are best possible.

Example 1. Let $X=L_p(\Gamma)$, $1\leqq p<\infty$, $p\neq 2$, where $\Gamma=\{\lambda\colon |\lambda|=1\}$. If $x\in X$, let $[T(t)x](e^{2\pi is})=x(e^{2\pi i(s+t)})$ for $-\infty<t<\infty$. This is a strongly continuous group of isometries with $T(1)=I$, but $T(t)$ is not scalar type or even spectral for irrational t as proved by Fixman [4]. This shows that we cannot remove the restrictions on the spectra, even if we have a group instead of a semigroup.

Example 2. Let $X=L_2[0, 1]$. For every $t\geqq 0$ and $x\in X$, let

$$[T(t)x](s) = \begin{cases} f(t+s), & t+s\leqq 1 \\ 0 & t+s>1. \end{cases}$$

$T(1)=0$ is scalar type, but $T(t)$ is scalar type for no t in the interval (0, 1) since it is nonzero nilpotent. This shows that in Theorem 1 we cannot do without the condition that $T(1)$ is one-to-one, i.e., in Theorem 2 we cannot conclude that every $T(t)$ is scalar type, but only spectral, even when X is a Hilbert space. This shows, also, that in Theorem 3 we cannot replace "group" by "semigroup".

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