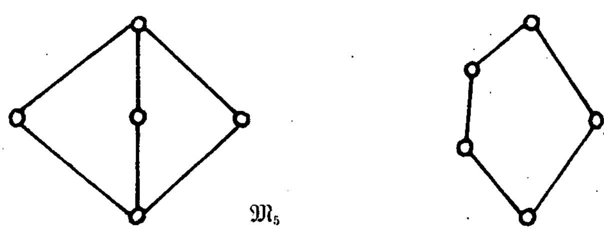
Equational classes which cover the class of distributive lattices

By E. FRIED in Budapest

Introduction. It is well-known that there are exactly two equational classes of lattices which cover the class of all distributive lattices. These are the classes generated by $\mathfrak{M}_5$ and by $\mathfrak{N}_5$ (see Fig. 1) and one of them is always contained in any equational class of lattices containing properly the class of all distributive lattices.



$\mathfrak{M}_5$ $\mathfrak{N}_5$

Fig. 1

The class of *weakly associative lattices* (in short: WALs) contains two other equational classes which also cover the class of all distributive lattices.

It has been a conjecture since the introduction of WALs in 1970, that there are no other equational classes of WALs which cover the class of all distributive lattices. The aim of this paper is to prove the existence of an equational class of WALs different from the class of all distributive lattices which does not contain the four equational classes in question.

Preliminaries. An algebra $\mathfrak{A}=\langle A; \vee, \wedge\rangle$ is a WAL if the two binary operations are idempotent and commutative and over the two absorption identities the two weak associativities hold:

$$\{(x\vee z)\wedge(y\vee z)\}\wedge z = \{(x\wedge z)\vee(y\wedge z)\}\vee z = z.$$

The relations $x=x\wedge y$ and $y=x\vee y$ are equivalent and will be denoted by $x\leqq y$ ($x<y$ means $x\leqq y$ and $x\neq y$). The relation $\leqq$ is reflexive and antisymmetrical and

$x \vee y$ and $x \wedge y$ are the least upper bound and the greatest lower bound of x and y, respectively. In such a way one can define WALs just as lattices in the transitive case. We shall denote $a < b$ by an arrow which goes from a to b.

Fig. 2 shows two WALs both of which are subdirectly irreducible. They are typical, for they show the two possibilities when $a < b < c$ does not imply $a < c$. It is not to hard to prove that both of the equational classes generated by them cover the class of all distributive lattices.

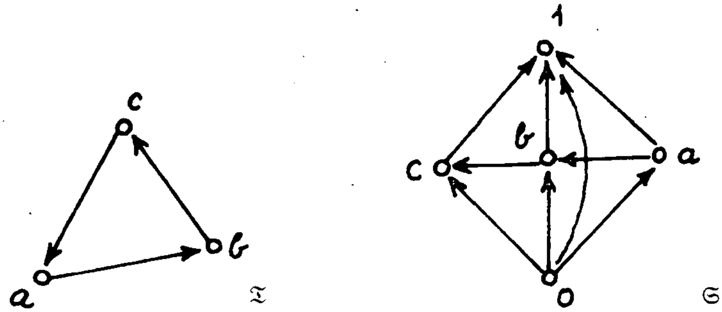


Fig. 2

A WAL $\mathfrak{A} = \langle A; \vee, \wedge \rangle$ has the unique bound property (further on UBP) if, for different $a, b \in A$, $a \leqq c$ and $b \leqq c$ imply $c = a \vee b$, and $d \leqq a$ and $d \leqq b$ imply $d = a \wedge b$. It was proven in [2] that $\mathfrak{A}$ has the UBP if and only if it is subdirectly irreducible and it satisfies the congruence extension property if and only if each subalgebra of $\mathfrak{A}$ is a simple one.

The construction of the desired class. For the construction we need two propositions.

Proposition 1. *There exists a* WAL *having the* UBP *and containing none of* $\mathfrak{M}_5$, $\mathfrak{N}_5$, $\mathfrak{T}$ *and* $\mathfrak{S}$.

Proof. Let Q be the field of rationals and let α be a zero of the irreducible polynomial $f(x) = x^3 + 3x + 1$. Let us denote by $a\alpha^2 + b\alpha + c$ the element (a, b, c) of the two-dimensional projective plane over Q $(a, b, c \in Q)$. Let, for $\beta, \gamma \in Q(\alpha)$, $\beta > \gamma$ mean the existence of a linear polynomial $r(x)$ over Q such that $\beta = \gamma \cdot r(\alpha)$. It was proven in [3] that in this manner we arrive at a WAL $\mathfrak{A}$ satisfying UBP. Thus, $\mathfrak{A}$ cannot contain a three-element chain, i.e., $\mathfrak{A}$ does not contain any of $\mathfrak{M}_5$, $\mathfrak{N}_5$, and $\mathfrak{S}$. $f'(x) = 3x^2 + 3 > 0$ implies that $x^3 + 3x + p$ has for no rational p two, hence three, rational roots. Let, now, $\beta < \gamma < \delta < \varepsilon$ in $Q(\alpha)$, i.e., $\varepsilon = \beta \cdot r_1(\alpha) \cdot r_2(\alpha) \cdot r_3(\alpha)$, where the $r_i(x)$-s are linear polynomials over Q. As $x^3 + 3x + p$ is never a product of three linear factors over Q the element $r_1(\alpha) \cdot r_2(\alpha) \cdot r_3(\alpha)$ does not belong to Q.

Hence, β and ε are different elements of the projective plane, i.e., $\mathfrak{A}$ does not contain $\mathfrak{T}$, either.

Remark. This method does not work more for finite fields. Let a, b, and c be elements of the finite field K. The polynomial $f(x)=x^3+ax^2+bx+c$ orders to each element u of K the element $f(u)$ of K. If $f(u)$ runs over K then $f(x)+v$ is for no v in K irreducible. When the method gives us a WAL then $f(x)$ must be irreducible. Hence, in this case there must exist different u_1, u_2 in K such that $f(u_1)=f(u_2)=v$. Thus, the polynomial $f(x)-v$ is the product of three linear polynomials, say $r_1(x)$, $r_2(x)$, $r_3(x)$, i.e., for the zero α of $f(x)$ we have $1<r_1(\alpha)<r_1(\alpha)\cdot r_2(\alpha)<r_1(\alpha)\cdot r_2(\alpha)\cdot r_3(\alpha)=v$. Since 1 and v are the same points of the projective plane this WAL must contain $\mathfrak{T}$.

Let $\mathfrak{A}$ be a WAL satisfying UBP and not containing any subalgebra isomorphic to $\mathfrak{T}$. These two properties mean that for the elements $a<b<c$ in $\mathfrak{A}$ the elements a and c are incomparable, i.e., neither $a\leqq c$ nor $c\leqq a$ are valid. Such a WAL we shall call a *scattering* WAL.

Proposition 2. *Homomorphic images, subalgebras and primeproducts of scattering* WALs *are scattering, too.*

Proof. The first statement is implied by the simpleness of scattering WALs. The second statement is obvious. Since the class of scattering WALs is a first order class, the third statement is also valid.

Theorem. *The equational class generated by the scattering* WALs *does not contain any of the given four classes.*

Proof. It was proven in [1] that the congruence-lattice of any WAL is a distributive one. The homomorphic images of subalgebras of primeproducts of scattering WALs are, by Proposition 2, scattering WALs. Thus, applying the well-known result of Jónsson (Theorem 3.3 in [5]) we have that the equational class in question does not contain other subdirectly irreducible WALs. Since $\mathfrak{M}_5$, $\mathfrak{N}_5$, $\mathfrak{T}$ and $\mathfrak{S}$ are subdirectly irreducible the theorem is proven.[1]

Problems. The theorem, of course, does not mean the existence of a new equational class covering the class of distributive lattices. It is possible that this class is not covered by any equational subclass of the constructed above. We state some problems concerning this question.[2]

---

[1]) This proof was shortened by W. A. Lampe in Honolulu.

[2]) In the meantime, Problem 1 was solved by R. Freese in Honolulu. The lattice of the equational classes of WALs is dual to an algebraic lattice. Since the class of distributive lattices is finitely based this class is a compact element of the lattice of classes. Thus, each maximal chain between the class of all distributive lattices and of all scattering WALs contains an element which covers the class of all distributive lattices. Hence, the existence of a fifth covering class is proven.

1. Is there any scattering WAL $\mathfrak{A}$ having no nontrivial subalgebra which is not isomorphic to $\mathfrak{A}$?

2. Are there any finite scattering WALs? (The existence of such a WAL would imply that the answer of problem 1 is affirmative.)

3. Is there any equational class of WALs different from the class of all distributive lattices which does not contain either scattering WALs or any of the given four WALs?

References

[1] E. Fried, Tournaments and nonassociative lattices, *Ann. Univ. Sci. Budapest*, Sect. Math., 13 (1970), 151—164.

[2] E. Fried, Subdirect irreducible weakly associative lattices with congruence extension property, *Ann. Unio. Sci Budapest*, Sect. Math., **17** (1974), 59—68.

[3] E. Fried—V. T. Sós, Weakly associative lattices and projective planes *Algebra Universalis*, **5** (1975), 62—67.

[4] G. Grätzer, *Lattice theory, first concepts and distributive lattices*, Freeman and Co. (1971).

[5] B. Jónsson, Algebras whose congruence lattices are distributive, *Math. Scand.*, **21** (1967), 110—121.

(Received April 24, 1974)