Modular lattices of locally finite length

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A lattice L is said to be of locally finite length if each bounded chain in L is finite. A sublattice S of L will be called a c-sublattice if, whenever $s_1, s_2 \in S$ and s_1 covers s_2 in S, then s_1 covers s_2 in L.

It is well-known that a lattice is modular if and only if it does not contain a sublattice isomorphic to the lattice A on Fig. 1 (this result goes back to DEDEKIND [3]; cf. also [1], [4], [6]). A relatively complemented lattice with the greatest element 1 and the least element 0 is modular if and only if it does not contain a sublattice S isomorphic to the lattice on Fig. 1 such that $0 \in S$ and $1 \in S$ (SZÁSZ [7]). A finite lattice is nonmodular if and only if it contains a lattice on Fig. 1 as a sublattice such that a covers b (cf. GRÄTZER [4], p. 151). Other conditions for a lattice to be modular were established by CROISOT [2].

The following result on finite modular lattices is known (cf. GRÄTZER [4], p. 151):

(∗) Let L be a finite modular lattice. Then L is nondistributive if and only if it contains the lattice on Fig. 4 as a sublattice such that a, b, and c cover u and v covers a, b, and c.

Thus distributive lattices in the class of finite modular lattices can be characterized by means of c-sublattices.

The purpose of this note is to characterize modular lattices in the class of lattices of locally finite length by means of their c-sublattices. A nonmodular lattice of locally finite length need not contain a c-sublattice isomorphic to the lattice A on Fig. 1. Let B be the lattice on Fig. 2 and let B' be the lattice dual to B. We denote by $L(m, n)$ the lattice on Fig. 3 ($m \geqq 3$, $n \geqq 4$). Further let C be the lattice on Fig. 4. The following theorems will be proved:

Theorem 1. *Let L be a lattice of locally finite length. Then L is modular if and only if L does not contain a sublattice isomorphic to some of the following lattices: B, B', $L(m, n)$ ($m \geqq 3$, $n \geqq 4$).*

Corollary. (ŠIK [5].) *A lattice of locally finite length fulfilling the upper covering condition is modular if and only if it does not contain a sublattice isomorphic to B.*

Theorem 2. *Let L be a lattice of locally finite length. Then L is distributive if and only if it does not contain a c-sublattice isomorphic with some of the following lattices: B, B', $L(m, n)(m \geqq 3,\ n \geqq 4)$, C.*

The standard terminology of the lattice theory will be used (cf. [1], [4], [6]). The lattice operations will be denoted by $\wedge$, $\vee$. Let L be a lattice, $a, b \in L$, $a \leqq b$. The interval $[a, b]$ is the set $\{x \in L: a \leqq x \leqq b\}$. If $a < b$ and $[a, b] = \{a, b\}$, then $[a, b]$ is called a prime interval; in this case b covers a (and a is covered by b).

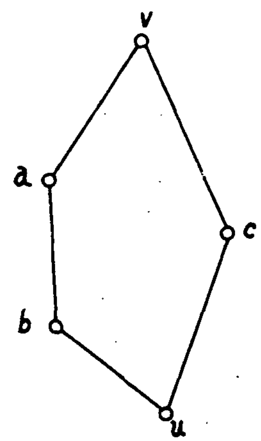

Fig. 1

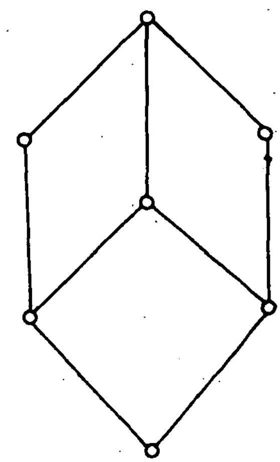
Fig. 2

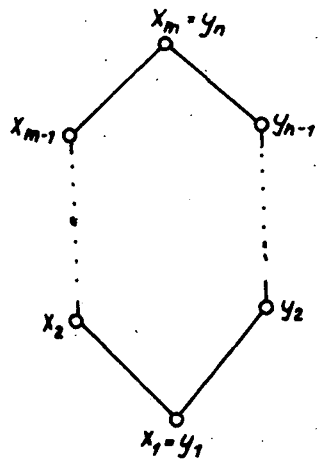

Fig. 3

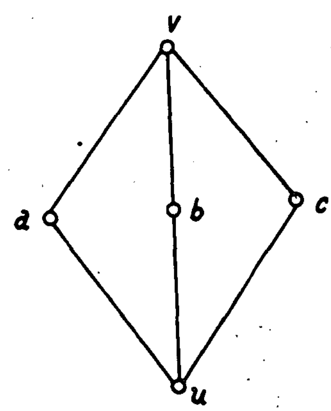

Fig. 4

If two elements x, y of a lattice are uncomparable, we write $x|y$. Let m, n be positive integers, $m \geqq 3$, $n \geqq 4$. We denote by $L(m, n)$ a lattice with elements $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$ such that $x_1 = y_1$, $x_m = y_n$, $x_i < x_{i+1}$ $(i = 1, \ldots, m-1)$, $y_j < y_{j+1}$ $(j = 1, \ldots, n-1)$, $x_i | y_j$ $(i = 2, 3, \ldots, m-1;\ j = 2, 3, \ldots, n-1)$ (cf. Fig. 3).

Let L be a lattice of locally finite length. We denote by M_1 the set of all intervals $[u, v]$ of L such that there are elements $a, b \in [u, v]$, $a|b$ fulfilling the conditions:

(i) both a and b are covered by v;

(ii) $u = a \wedge b$ and either a or b does not cover u.

Let M_2 be defined dually and put $M = M_1 \cup M_2$. The set M is partially ordered by the inclusion. Since L is of locally finite length, M satisfies the descending chain condition. If L is nonmodular, then we have $M \neq \emptyset$ and hence the set M_0 of all minimal elements of M is nonempty.

Let us recall that if K is a bounded lattice of locally finite length and if L is modular, then any two maximal chains in K have the same number of elements (cf. [1]).

Proof of Theorem 1.

The lattices B, B', $L(m, n)$ $(m \geqq 3, n \geqq 4)$ being nonmodular, it suffices to verify the assertion "only if".

Assume that L is nonmodular. Then $M_0 \neq \emptyset$. Let $[u, v]$ be a fixed element of M_0. We may suppose that $[u, v] \in M_1$ (in the case $[u, v] \in M_2$ we would apply a dual method). Let a, b be as in (i) and (ii).

Let $u = x_1 < \ldots < x_{m-1} = a$, $u = y_1 < \ldots < y_{n-1} = b$ be two maximal chains in $[u, a]$, $[u, b]$, respectively. In case $x_2 \vee y_2 = v$ the set $N_1 = \{u, v, x_2, \ldots, x_{m-1}, y_2, \ldots, y_{n-1}\}$ is a c-sublattice isomorphic to $L(m, n)$, $m \geqq 3$, $n \geqq 3$ and by (ii), either m or n is $\geqq 4$. Therefore N_1 is isomorphic to one of the lattices listed in the Theorem.

Suppose that $x_2 \vee y_2 = v_1 < v$. Then $[u, v_1]$ is a proper subset of $[u, v]$, both x_2 and y_2 cover u, thus with respect to the minimality of $[u, v]$ in M it follows, that v_1 covers both x_2 and y_2, as well. Obviously $x_2 | b$ and $y_2 | a$. Therefore

$$(1)\quad v_1 \vee a = v_1 \vee b = v, \qquad (2)\quad v_1 \wedge a = x_2, \quad v_1 \wedge b = y_2, \qquad (3)\quad x_2 \vee b = y_2 \vee a = v.$$

From (1)—(3) it follows that the set $N_2 = \{a, b, u, v, x_2, y_2, v_1\}$ is a sublattice of L isomorphic to B.

From the minimality of $[u, v]$ it follows that the lattices $[x_2, v]$ and $[y_2, v]$ are modular. Let $\bar{v} \in [v_1, v]$ such that $\bar{v}$ covers v_1. Let $\bar{a} = \bar{v} \wedge a$, $\bar{b} = \bar{v} \wedge b$.

Because of the modularity of $[x_2, v]$ and $[y_2, v]$ both $\bar{a}$ and $\bar{b}$ are covered by $\bar{v}$, furthermore $\bar{a} | \bar{b}$, $\bar{a} \wedge \bar{b} = u$ and neither $\bar{a}$ nor $\bar{b}$ covers u. Hence $[u, \bar{v}] \in M$ and $[u, \bar{v}] \subseteqq [u, v]$, i.e., $[u, \bar{v}] = [u, v]$; $\bar{v} = v$.

Thus we proved that v covers v_1; therefore a covers x_2 and b covers y_2 which proves that N_2 is a c-sublattice. Q.e.d.

Lemma. *Let L be a non-distributive modular lattice fulfilling the descending chain condition. Then L contains a c-sublattice isomorphic to C.*

Remark. Since a distributive lattice can not contain any sublattice isomorphic to C, this Lemma generalizes the statement $(*)$ to modular lattices fulfilling the descending chain condition.

Proof of the Lemma. In fact, $C=\{u, a, b, c, v\}$ $(u \leqq a, b, c \leqq v)$ is a sublattice of L. Let $\bar{a} \in [u, a]$ such that $\bar{a}$ covers u. Set

$$\bar{v} = (\bar{a} \vee b) \wedge (\bar{a} \vee c), \quad \bar{b} = b \wedge (\bar{a} \vee c), \quad \bar{c} = c \wedge (\bar{a} \vee b).$$

Clearly $\bar{b} = \bar{v} \wedge b$ and $\bar{c} = \bar{v} \wedge c$. Using the projectivity it follows easily that all intervals $[\bar{a}, \bar{v}]$, $[\bar{b}, \bar{v}]$, $[\bar{c}, \bar{v}]$, $[u, \bar{b}]$, $[u, \bar{c}]$ are prime. From this we obtain that the set $C = \{u, \bar{a}, \bar{b}, \bar{c}, \bar{v}\}$ is a c-sublattice of L isomorphic to C. The proof of Theorem 2 follows immediately from Lemma and Theorem 1.

Added in proof. Theorem 1 can be deduced also from Thm. 2.2 of V. Vilhelm, Двойственное себе ядро условий Биркгофа в структурах с конечными цепями, *Czech. Math. J.*, **5** (1955), 439—450.

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