On locally regular Rees matrix semigroups

By LÁSZLÓ MÁRKI in Budapest

The aim of this paper is to investigate the behaviour of locally regular Rees matrix semigroups over a semigroup with zero and identity with respect to certain properties. This class of semigroups was defined by STEINFELD [3] in the following way.

Let H be a semigroup with zero 0 and identity e. Let $M^\circ = M^\circ(H; I, \Lambda; P)$ denote the Rees matrix semigroup over H with sandwich matrix $P=(p_{\lambda i})$ ($\lambda \in \Lambda$; $i \in I$; $p_{\lambda i} \in H$). Denote the elements of M° by $(a)_{i\lambda}$ with a in H, i in I, and λ in Λ. The product of the matrices $(a)_{i\lambda}$, $(b)_{j\mu}$ is defined by

$$(a)_{i\lambda} \circ (b)_{j\mu} = (a p_{\lambda j} b)_{i\mu} \quad (a, b \in H;\ i, j \in I;\ \lambda, \mu \in \Lambda).$$

We say that $M^\circ(H; I, \Lambda; P)$ is *locally regular* if $P=(p_{\lambda i})$ has the following properties:

1) in every row λ of P there exists an element $p_{\lambda j(\lambda)}$ $\left(j(\lambda) \in I\right)$ which has a right inverse $p'_{\lambda j(\lambda)}$ in H, that is,

$$p_{\lambda j(\lambda)} p'_{\lambda j(\lambda)} = e;$$

2) in every column i of P there exists an element $p_{\mu(i) i}$ $\left(\mu(i) \in \Lambda\right)$ which has a eft inverse $p''_{\mu(i) i}$ in H, that is,

$$p''_{\mu(i) i} p_{\mu(i) i} = e;$$

3) there exists at least one element $p_{\lambda i}$ in P which has a right and left inverse in H.

One can see immediately that a Rees matrix semigroup over a group with zero is locally regular if and only if it is regular, hence, by the Rees representation theorem, if and only if it is completely 0-simple, which means that an abstract characterization of the class of locally regular Rees matrix semigroups yields a generalization of the Rees theorem. This characterization was given by STEINFELD [3] by means of the notion of similarity of one-sided ideals of a semigroup, introduced in the same paper of his.

The left ideals L_1 and L_2 of a semigroup S are said to be *left similar* if there exists a one-to-one mapping φ of L_1 onto L_2 such that $(sx)\varphi = s(x\varphi)$ for all $s \in S$ and $x \in L_1$. If, in addition, we have $x\varphi \in xS$ and $y\varphi^{-1} \in yS$ for all $x \in L_1$ and $y \in L_2$, then we say that L_1 and L_2 are *strongly left similar* (this notion proved to be useful in [1], where it was also shown that left similarity and strong left similarity of the left ideals L_1 and L_2 are equivalent in the case when L_1 and L_2 both can be generated by regular elements). Dually one defines right similarity and strong right similarity of right ideals of a semigroup S. Let S be a semigroup with 0 such that

$$S = \bigcup_{\lambda \in \Lambda} Se_\lambda = \bigcup_{i \in I} e_i S \quad (e_\lambda^2 = e_\lambda;\ e_i^2 = e_i;\ I \cap \Lambda \neq \varnothing)$$

where Se_λ $(\lambda \in \Lambda)$ $[e_i S$ $(i \in I)]$ are (strongly) left [right] similar left [right] ideals of S with $Se_\mu \cap Se_\nu = 0$ $(\mu, \nu \in \Lambda;\ \mu \neq \nu)$ and $e_j S \cap e_k S = 0$ $(j, k \in I;\ j \neq k)$. We call a semigroup with these properties *similarly decomposable.*

Now Theorem 4.1 of STEINFELD [3] asserts that a semigroup issimilarly decomposable if and only if it is isomorphic to a locally regular Rees matrix semigroup over a semigroup with zero and identity.

As far as we know, regularity and simplicity properties of similarly decomposable semigroups have not been studied yet. We are going to show that a similarly decomposable semigroup can have, but need not have such properties. To be more precise, we shall see that several properties (regularity, 0-simplicity, 0-bisimplicity, complete 0-simplicity) of a locally regular Rees matrix semigroup $M^\circ = M^\circ(H; I, \Lambda; P)$ depend solely upon the underlying semigroup H, while other properties (like inversity, semi-simplicity, left- or right- or intra-regularity) depend also upon the sandwich matrix P of M°. However, of the latter properties we shall investigate inversity only, for the other ones we but mentioned that the fact that M° has any of them depends on H and P (and I and Λ) either.

In the sequel, H will always denote a semigroup with zero 0 and identity e and $M^\circ = M^\circ(H; I, \Lambda; P)$ a locally regular Rees matrix semigroup over H.

From the multiplication law $\circ$ of M° it follows immediately that for any $i \in I$ and $\lambda \in \Lambda$, the set $\{(a)_{i\lambda} \mid a \in H\}$ endowed with the multiplication $\circ$ forms a subsemigroup $M_{i\lambda}^\circ$ of M°. It is well-known from the theory of completely 0-simple semigroups, that if H is a group with zero, then each $M_{i\lambda}^\circ$ is either a zero-semigroup or is isomorphic to H. The following two lemmas (the first of which includes the above mentioned result) will show that this is far from being true in general.

Lemma 1. *$M_{i\lambda}^\circ$ is isomorphic to H if and only if the entry $p_{\lambda i}$ of the sandwich matrix P has a (two-sided) inverse (in H).*

Proof. Suppose that $\varphi\colon H \to M_{i\lambda}^\circ\colon a \to (\varphi'(a))_{i\lambda}$ is an isomorphism, then we have $\varphi(a) = \varphi(ea) = \varphi(e) \circ \varphi(a)$; putting $a = \varphi^{-1}((e)_{i\lambda})$, herefrom we obtain that $\varphi'(e) p_{\lambda i} = e$. Similarly, $\varphi(a) = \varphi(a) \circ \varphi(e)$ implies $p_{\lambda i} \varphi'(e) = e$, thus $\varphi'(e)$ is an inverse of $p_{\lambda i}$.

Suppose now that $p_{\lambda i}$ has an inverse $p'_{\lambda i}$. We shall show that

$$\varphi\colon H \to M_{i\lambda}^\circ\colon a \to (a p'_{\lambda i})_{i\lambda}$$

is an isomorphism.

In fact,

1) φ is a homomorphism since for any $a, b \in H$ we have

$$\varphi(ab) = (abp'_{\lambda i})_{i\lambda} = (ap'_{\lambda i}p_{\lambda i}bp'_{\lambda i})_{i\lambda} = (ap'_{\lambda i})_{i\lambda} \circ (bp'_{\lambda i})_{i\lambda} = \varphi(a) \circ \varphi(b),$$

2) φ is one-to-one as $ap'_{\lambda i}=bp'_{\lambda i}$ implies $a=ap'_{\lambda i}p_{\lambda i}=bp'_{\lambda i}p_{\lambda i}=b$,

3) φ is onto since for any $a \in H$ we have $(a)_{i\lambda}=(ap_{\lambda i}p'_{\lambda i})_{i\lambda}=\varphi(ap_{\lambda i})$, q.e.d.

Corollary. *If the entry $p_{\lambda i}$ of the sandwich matrix P has a one-sided inverse which is not a two-sided inverse then $M^\circ_{i\lambda}$ is neither isomorphic to H nor is it a zero semigroup.*

Note that for 1) and 2) we used only that $p'_{\lambda i}$ is a left inverse of $p_{\lambda i}$, thus φ is a monomorphism in this case already. Similarly, if $p'_{\lambda i}$ is a right inverse of $p_{\lambda i}$, then

$$\tilde{\varphi}: a \to (p'_{\lambda i}a)_{i\lambda}$$

is a monomorphism of H into $M^\circ_{i\lambda}$. This remark constitutes our

Lemma 2. *If $p_{\lambda i}$ has a one-sided inverse, $M^\circ_{i\lambda}$ contains a subsemigroup which is isomorphic to H.*

Theorem 1. *$M^\circ = M^\circ(H; I, \Lambda; P)$ is 0-simple if and only if H is 0-simple.*

Proof. Suppose that M° is 0-simple and let a and b be arbitrary non-zero elements of H, we have to show the existence of elements $x, y \in H$ with $xay=b$. Now choose arbitrarily $i, j \in I$ and $\lambda, \mu \in \Lambda$, then, by the 0-simplicity of M°, there exist indices $k \in I$, $\nu \in \Lambda$ and elements $x', y' \in H$ such that

$$(x')_{j\nu} \circ (a)_{i\lambda} \circ (y')_{k\mu} = (b)_{j\mu},$$

that is,

$$x'p_{\nu i}ap_{\lambda k}y' = b.$$

Thus we have $xay=b$ with $x=x'p_{\nu i}$ and $y=p_{\lambda k}y'$.

Suppose now that H is 0-simple and let $(a)_{i\lambda}$ and $(b)_{j\mu}$ be arbitrary non-zero elements of M°. Since H is 0-simple, there exist elements $x', y' \in H$ with $x'ay'=b$. Now let $\nu(i) \in \Lambda$ and $k(\lambda) \in I$ be indices for which $p_{\nu(i)i}$ has a left inverse $p''_{\nu(i)i}$ and $p_{\lambda k(\lambda)}$ has a right inverse $p'_{\lambda k(\lambda)}$, and put $x=x'p''_{\nu(i)i}$, $y=p'_{\lambda k(\lambda)}y'$. Then we have

$$(x)_{j\nu(i)} \circ (a)_{i\lambda} \circ (y)_{k(\lambda)\mu} = (xp_{\nu(i)i}ap_{\lambda k(\lambda)}y)_{j\mu} = (x'ay')_{j\mu} = (b)_{j\mu},$$

q.e.d.

Theorem 2. *$M^\circ = M^\circ(H; I, \Lambda; P)$ is completely 0-simple if and only if H is completely 0-simple.*

Proof. If H is completely 0-simple, it is a group with zero (since by Rees [2], a completely 0-simple semigroup with identity is a group with zero). Then local regularity of M° is equivalent to regularity by one of our first remarks, thus M° is completely 0-simple by the Rees representation theorem (s. [2]).

Suppose now that M° is completely 0-simple, then H is 0-simple by Theorem 1. On the other hand, by the Corollary of Lemma 1, H is isomorphic to a subsemigroup of M°, thus all idempotents of H are primitive. Hence H is completely 0-simple.

Theorem 3. *$M^\circ = M^\circ(H; I, \Lambda; P)$ is 0-bisimple if and only if H is 0-bisimple.*

Proof. We shall see that the $\mathscr{R}$-classes of M° are of the form

$$\{(a)_{i\lambda} | a \in R, \lambda \in \Lambda\}$$

where i is an element of I and R is an $\mathscr{R}$-class of H. Since the same proof gives a similar form for the $\mathscr{L}$-classes of M°, this implies that the $\mathscr{D}$-classes of M° are exactly the sets of the form

$$\{(a)_{i\lambda} | a \in D, i \in I, \lambda \in \Lambda\}$$

where D is a $\mathscr{D}$-class of H, which proves our assertion.

Let $a, b \in H$ with $a\mathscr{R}b$, $i \in I$ and $\lambda, \mu \in \Lambda$, we are going to show that $(a)_{i\lambda}\mathscr{R}(b)_{i\mu}$ in M°. From $a\mathscr{R}b$ it follows that there exist elements $x', y' \in H$ with $a = bx'$ and $b = ay'$. Now let $j(\lambda) \in I$ and $k(\mu) \in I$ be indices for which $p_{\lambda j(\lambda)}$ and $p_{\lambda k(\mu)}$ have right inverses $p'_{\lambda j(\lambda)}$ and $p'_{\mu k(\mu)}$, respectively, and put $x = p'_{\mu k(\mu)} x'$, $y = p'_{\lambda j(\lambda)} y'$, then we have

$$(b)_{i\mu} \circ (x)_{k(\mu)\lambda} = (bp_{\mu k(\mu)} x)_{i\lambda} = (bx')_{i\lambda} = (a)_{i\lambda}$$

and similarly $(a)_{i\lambda} \circ (y)_{j(\lambda)\mu} = (b)_{i\mu}$. Thus $(a)_{i\lambda}\mathscr{R}(b)_{i\mu}$.

We still have to show that $(a)_{i\lambda}\mathscr{R}(b)_{j\mu}$ in M° implies $i = j$ and $a\mathscr{R}b$ in H. In fact, if $(a)_{i\lambda}\mathscr{R}(b)_{j\mu}$, then there exist elements $(x)_{k\nu}$ and $(y)_{l\pi}$ with

$$(a)_{i\lambda} \circ (x)_{k\nu} = (b)_{j\mu} \quad \text{and} \quad (b)_{j\mu} \circ (y)_{l\pi} = (a)_{i\lambda},$$

that is,

$$(ap_{\lambda k} x)_{i\nu} = (b)_{j\mu} \quad \text{and} \quad (bp_{\mu l} y)_{j\pi} = (a)_{i\lambda}$$

which imply, among others, $i = j$, $a(p_{\lambda k} x) = b$ and $b(p_{\mu l} y) = a$. The last two equations give $a\mathscr{R}b$ in H, which completes the proof of our theorem.

Theorem 4. *$M^\circ = M^\circ(H; I, \Lambda; P)$ is regular if and only if H is regular.*

Proof. Let H be regular, $(a)_{i\lambda}$ be an arbitrary element of M° and $j(\lambda) \in I$, $\mu(i) \in \Lambda$ be indices for which $p_{\lambda j(\lambda)}$ has a right inverse $p'_{\lambda j(\lambda)}$ and $p_{\mu(i) i}$ has a left inverse $p''_{\mu(i) i}$. By the regularity of H we have $a = aya$ with some $y \in H$, thus we also have

$$(a)_{i\lambda} = (ap_{\lambda j(\lambda)} p'_{\lambda j(\lambda)} y p''_{\mu(i) i} p_{\mu(i) i} a)_{i\lambda} = (a)_{i\lambda} \circ (p'_{\lambda j(\lambda)} y p''_{\mu(i) i})_{j(\lambda)\mu(i)} \circ (a)_{i\lambda}$$

which proves the regularity of M°.

On the other hand, suppose that M° is regular, let a be an arbitrary element of H and choose any indices $i\in I$ and $\lambda\in\Lambda$. Then we have

$$(a)_{i\lambda}\circ(x)_{j\mu}\circ(a)_{i\lambda}=(a)_{i\lambda}$$

for some $x\in H$, $j\in I$ and $\mu\in\Lambda$, hence

$$a=a(p_{\lambda j}xp_{\mu i})a.$$

Thus H is also regular.

Remark. Combining the first part of this proof with the fact that if $a=axa$ then xax is a generalized inverse of a, we obtain that each element of $M^\circ_{i\lambda}$ in a regular M° has a generalized inverse in $M^\circ_{j(\lambda)\mu(i)}$.

The only if parts of Theorems 1 and 4, in the proofs of which not even local regularity of M° was made use of, were already given in the most general case by VENKATESAN [4]. Corollary 1 to Proposition 1 and Corollary 1 to Theorem 1 of VENKATESAN [4], together with our Theorem 4, give the following result for locally regular Rees matrix semigroups:

Theorem 5. *$M^\circ=M^\circ(H;I,\Lambda;P)$ is a union of its completely 0-simple ideals if and only if the same is true for H.*

In connection with the Corollary to Lemma 2, we should like to mention that the behaviour of the subsemigroups $M^\circ_{i\lambda}$ of M° is, in general, far from being so nice as that of H with respect to the above treated properties. As an illustration, let us see the following example:

Let H be the bicyclic semigroup $\mathscr{C}(q,r)$ with zero adjoined, $I=\Lambda=\{0,1,2,\dots\}$ and the sandwich matrix P be

$$P=\begin{array}{c}
\begin{array}{cccccc} \underbrace{0} & \underbrace{1} & \underbrace{2} & \underbrace{3}\ \dots & \underbrace{m} & \dots \end{array}\\
\begin{pmatrix}
e & 0 & 0 & 0\ \dots & 0 & \dots\\
0 & r & q & r^3\ \dots & r^m & \dots\\
0 & q^2 & r^2 & 0\ \dots & 0 & \dots\\
0 & q^3 & 0 & 0\ \dots & 0 & \dots\\
\vdots & \vdots & \vdots & \vdots & & \\
0 & q^n & 0 & 0 & 0 & \\
\vdots & \vdots & \vdots & \vdots & &
\end{pmatrix}
\end{array},$$

and consider the Rees matrix semigroup $M^\circ(H;I,\Lambda;P)$. Since for all $n\geqq 0$, r^n has a left inverse and q^n has a right inverse, M° is a locally regular Rees matrix semigroup. It is well-known that this H is 0-bisimple, thus the same is true for M°. If $n\geqq 3$, by Lemma 2, M°_{n1} contains a subsemigroup which is isomorphic to H, however, M°_{n1} is not even regular. In fact, let $k<n$ and consider the element $(r^kq^l)_{n1}$ with some $l\geqq 0$, then for any element $(r^sq^t)_{n1}$ of M°_{n1} we have

$$(r^kq^l)_{n1}\circ(r^sq^t)_{n1}\circ(r^kq^l)_{n1}=(r^kq^lq^nr^sq^tq^nr^kq^l)_{n1}=(r^kq^{l+n}r^sq^{t+n-k+l})_{n1}\neq(r^kq^l)_{n1}$$

since multiplication by $r^k q^{t+n} r^s$ from the left cannot reduce the exponent $t+n-k+l>l$ of q.

At last we give a necessary and sufficient condition that $M^\circ(H; I, \Lambda; P)$ be an inverse semigroup.

Theorem 6. *The locally regular Rees matrix semigroup $M^\circ = M^\circ(H; I, \Lambda; P)$ is an inverse semigroup if and only if the following conditions are satisfied: H is an inverse semigroup, in each row and each column of the sandwich matrix P there exists exactly one element which has a two-sided inverse (clearly, this implies $|I|=|\Lambda|$), and all the other entries of P are zero.*

Proof. Suppose first that $M^\circ(H; I, \Lambda; P)$ satisfies all these conditions, and, for any $\lambda \in \Lambda$ and $i \in I$, let $j(\lambda) \in I$ and $\mu(i) \in \Lambda$ denote the indices for which $p_{\lambda j(\lambda)}$ and $p_{\mu(i)i}$ have two-sided inverses.

Let $(a)_{i\lambda}$ and $(b)_{j\mu}$ be generalized inverses of each other, $a \neq 0$, then we have

$$a = ap_{\lambda j} bp_{\mu i} a \neq 0,$$

whence $j=j(\lambda)$ and $\mu=\mu(i)$. Suppose further that $(c)_{j(\lambda)\mu(i)}$ is also a generalized inverse of $(a)_{i\lambda}$. Then we have

$$(*) \qquad a = ap_{\lambda j(\lambda)} bp_{\mu(i)i} a = ap_{\lambda j(\lambda)} cp_{\mu(i)i} a,$$

$$(**) \qquad b = bp_{\mu(i)i} ap_{\lambda j(\lambda)} b, \quad c = cp_{\mu(i)i} ap_{\lambda j(\lambda)} c.$$

Multiplying the equations $(*)$ and $(**)$ from the right by $p_{\lambda j(\lambda)}$ and $p_{\mu(i)i}$, respectively, we obtain that $bp_{\mu(i)i}$ and $cp_{\mu(i)i}$ are both generalized inverses of $ap_{\lambda j(\lambda)}$ in H. In view of the inversity of H, this implies

$$bp_{\mu(i)i} = cp_{\mu(i)i},$$

and multiplication from the right by the inverse of $p_{\mu(i)i}$ gives now $b=c$. Hence each element of M° may have at most one generalized inverse, but it does have one, since by Theorem 4 the regularity of H implies that M° is also a regular semigroup. Thus M° is an inverse semigroup.

Conversely, suppose that M° is an inverse semigroup. By Theorem 4, H is regular, and as H is isomorphic to a subsemigroup of M° by the Corollary of Lemma 1, no element of H can have more than one generalized inverse element. Thus H is an inverse semigroup.

Let $(a)_{i\lambda}$ be an arbitrary non-zero element of M°, and suppose that $(b)_{j\mu}$ is the generalized inverse of $(a)_{i\lambda}$.

We have seen in the Remark after Theorem 4 that each $(a)_{i\lambda}$ has a generalized inverse in $M^\circ_{j(\lambda)\mu(i)}$, hence we must have, by the unicity of the generalized inverse element, $j=j(\lambda)$ and $\mu=\mu(i)$. On the other hand, $(b)_{j\mu}=(b)_{j(\lambda)\mu(i)}$ has $(a)_{i\mu}$ as its generalized inverse, but it also has a generalized inverse in $M^\circ_{j(\mu(i))\mu(j(\lambda))}$, whence

$ij = (\mu(i))$ and $\lambda = \mu(j(\lambda))$. Herefrom we can conclude that the elements $p_{\lambda j(\lambda)}$ and $p_{i\mu(i)}$ have (two-sided) inverses in H. Suppose now that the element $p_{\lambda m}$ has a right inverse $p'_{\lambda m}$ in H for some $m \in I$. Then we have

$$(a)_{i\lambda} = (a)_{i\lambda} \circ (b)_{j\mu} \circ (a)_{i\lambda} = (ap_{\lambda m}p'_{\lambda m}p_{\lambda j}bp_{\mu i}a)_{i\lambda} = (a)_{i\lambda} \circ (p'_{\lambda m}p_{\lambda j}b)_{m\mu} \circ (a)_{i\lambda}$$

and

$$b = bp_{\mu i}ap_{\lambda j}b = bp_{\mu i}ap_{\lambda m}p'_{\lambda m}p_{\lambda j}b,$$

multiplying here by $p'_{\lambda m}p_{\lambda j}$ from the left we obtain that

$$(p'_{\lambda m}p_{\lambda j}b)_{m\mu} = (p'_{\lambda m}p_{\lambda j}b)_{m\mu} \circ (a)_{i\lambda} \circ (p'_{\lambda m}p_{\lambda j}b)_{m\mu},$$

that is, $(p'_{\lambda m}p_{\lambda j}b)_{m\mu}$ is also a generalized inverse of $(a)_{i\lambda}$. Since $(a)_{i\lambda}$ may have but one generalized inverse, this implies $m = j = j(\lambda)$. In other words, in each row of P there exists exactly one element which has a right inverse in H, and we have seen that this element must have a two-sided inverse. Dually we obtain the analogous result for columns.

We still have to show that all the other entries of P are zero. Suppose that, on the contrary, there exists an entry $p_{\lambda i} \neq 0$ in P, which does not have an inverse of either sides in H. As H is an inverse semigroup, $p_{\lambda i}$ has a generalized inverse a in H:

$$ap_{\lambda i}a = a \quad \text{and} \quad p_{\lambda i}ap_{\lambda i} = p_{\lambda i}.$$

Then we also have
$$a = ap_{\lambda i}a = ap_{\lambda i}ap_{\lambda i}a$$
and
$$(a)_{i\lambda} = (ap_{\lambda i}ap_{\lambda i}a)_{i\lambda} = (a)_{i\lambda} \circ (a)_{i\lambda} \circ (a)_{i\lambda},$$

thus $(a)_{i\lambda}$ is a generalized inverse of itself, which contradicts the fact that M° is an inverse semigroup, since we have seen that in such a semigroup each element $(a)_{i\lambda}$ has its generalized inverse in $M^\circ_{j(\lambda)\mu(i)}$, and now $i \neq j(\lambda)$ for $p_{\lambda i}$ does not have a right inverse. This completes the proof of Theorem 6.

Remark. For the notions occurring in the following Corollary we refer to [3]. As it is easy to show that a 0-cancellative regular semigroup with identity is a group with zero adjoined, our Theorem 6 and Theorem 5.1 in [3] imply the following result:

Corollary. *For a special similarly decomposable semigroup S the following conditions are equivalent:*

(i) *S is regular,*
(ii) *S is an inverse semigroup,*
(iii) *S is completely 0-simple,*
(iv) *S is a Brandt semigroup.*

Thanks are due to Prof. O. Steinfeld for his attention during the preparation of the present work.

References

[1] L. Márki—O. Steinfeld, A generalization of Green's relations in semigroups, *Semigroup Forum*, 7 (1974), 74—85.

[2] D. Rees, On semi-groups, *Proc. Cambridge Phil. Soc.*, 36 (1940), 387—400.

[3] O. Steinfeld, On a generalization of completely 0-simple semigroups, *Acta Sci. Math.*, **28** (1967), 135—145.

[4] P. S. Venkatesan, Matrix semigroups over a semigroup with zero, *Math. Ann.*, **181** (1969), 60—64.

MATHEMATICAL INSTITUTE OF THE
HUNGARIAN ACADEMY OF SCIENCES
REÁLTANODA U. 13—15
H—1053 BUDAPEST

(Received January 16, 1974)