On a paper of Blum, Eisenberg, and Hahn concerning ergodic theory and the distribution of sequences in the Bohr group

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Let $\mathbf{Z}$ be the additive group of integers in the discrete topology, and let $\overline{\mathbf{Z}}$ be its Bohr compactification. In a recent paper, BLUM, EISENBERG, and HAHN [1] have pointed out a remarkable connection between the validity of the mean ergodic theorem for sums of the type $\frac{1}{N}\sum_{n=1}^{N} T^{a_n}f$, where (a_n), $n=1, 2, \dots$, is a given sequence of integers, and the distribution of the sequence (a_n) in $\overline{\mathbf{Z}}$. In fact, it is noted in that paper that the mean ergodic theorem holds for the above sums if and only if the sequence (a_n) is uniformly distributed in $\overline{\mathbf{Z}}$ in the sense of Definition 1 below. Therefore, it becomes an interesting problem to exhibit classes of sequences (a_n) that satisfy the required type of uniform distribution property. BLUM, EISENBERG, and HAHN have already made a contribution to this problem by providing a sufficient condition that can be checked fairly easily (see condition (i) in [1, p. 23] or condition (1) below). The authors state that they do not know of any sequence in $\mathbf{Z}$ that is uniformly distributed in $\overline{\mathbf{Z}}$ but does not satisfy the condition (1). It is the main purpose of this note to show that the condition (1) of BLUM, EISENBERG, and HAHN is certainly not necessary for uniform distribution in $\overline{\mathbf{Z}}$, and that one may in fact construct sequences in $\mathbf{Z}$ that are uniformly distributed in $\overline{\mathbf{Z}}$ but for which (1) fails drastically. At the same time, we exhibit large classes of sequences (a_n) in $\mathbf{Z}$ that are uniformly distributed in $\overline{\mathbf{Z}}$ and that can therefore be used to obtain generalized mean ergodic theorems.

We recall some well-known notions of uniform distribution in topological groups. For a detailed discussion of this topic, see KUIPERS and NIEDERREITER [2, Ch. 4]. Since all the groups we shall consider in the sequel will be abelian, we restrict our attention to this case.

This research was carried out while the author was supported by NSF grant GP 36418X1.

Definition 1. The sequence (x_n), $n=1, 2, \ldots$, in the locally compact abelian group G is called Hartman-uniformly distributed in G if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi(x_n) = 0$$

holds for all nontrivial characters χ of G.

Definition 2. The sequence (x_n), $n=1, 2, \ldots$, in the locally compact abelian group G is called uniformly distributed in G if for any subgroup H of G of compact index (i.e., for any closed subgroup H of G for which G/H is compact), the sequence (x_n+H), $n=1, 2, \ldots$, is Hartman-uniformly distributed in G/H.

A Hartman-uniformly distributed sequence in G is also uniformly distributed in G, but the converse is not true in general. For compact groups the two kinds of uniform distribution are the same. Moreover, the sequence (x_n) is Hartman-uniformly distributed in G if and only if (x_n) is uniformly distributed in the Bohr compactification $\bar{G}$ of G. For these results and for an exposition of the theory of uniform distribution in locally compact groups, see KUIPERS and NIEDERREITER [2, Ch. 4, Sect. 5].

For the special cases $G=\mathbf{Z}$ and $G=\mathbf{R}$, the additive group of real numbers in the usual topology, one arrives at the following equivalent characterizations by using the WEYL criterion for uniform distribution mod 1 (for details, see [2, Ch. 4, Sect. 5]).

Lemma 1. *The sequence* (x_n), $n=1, 2, \ldots$, *in* $\mathbf{R}$ *is uniformly distributed in* $\mathbf{R}$ *if and only if the sequence* $(x_n\alpha)$, $n=1, 2, \ldots$, *is uniformly distributed* mod 1 *for all nonzero real numbers* α.

Lemma 2. *The sequence* (a_n), $n=1, 2, \ldots$, *in* $\mathbf{Z}$ *is Hartman-uniformly distributed in* $\mathbf{Z}$ *if and only if* (a_n) *is uniformly distributed in* $\mathbf{Z}$ *and* $(a_n\alpha)$, $n=1, 2, \ldots$, *is uniformly distributed* mod 1 *for all irrational numbers* α.

We remark that the notion of uniform distribution in $\mathbf{Z}$ according to Definition 2 is identical with the notion introduced by NIVEN [3].

In the language of the present paper, the basic result mentioned in [1] concerning the mean ergodic theorem reads as follows: The mean ergodic theorem holds for sums of the type $\frac{1}{N}\sum_{n=1}^{N} T^{a_n}f$ if and only if the sequence (a_n), $n=1, 2, \ldots$, is Hartman-uniformly distributed in $\mathbf{Z}$. The sufficient condition for Hartman-uniform distribution in $\mathbf{Z}$ given by BLUM, EISENBERG, and HAHN is as follows. Let (a_n), $n=1, 2, \ldots$, be a sequence in $\mathbf{Z}$, let E_N be the set consisting of the first N terms of (a_n), and let E_N+k be the set E_N shifted by the integer k. Then, if

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{card}(E_N \cap (E_N+k)) = 1 \quad \text{for all} \quad k \in \mathbf{Z}, \tag{1}$$

the sequence (a_n) is Hartman-uniformly distributed in $\mathbf{Z}$ (see [1, Theorem 1]). Obviously, it suffices to consider only positive integers k in (1).

The following theorem provides many examples of Hartman-uniformly distributed sequences in $\mathbf{Z}$, most of which do not satisfy condition (1).

Theorem 1. *If (x_n), $n=1, 2, \ldots$, is uniformly distributed in $\mathbf{R}$, then the sequence $([x_n])$, $n=1, 2, \ldots$, of integral parts is Hartman-uniformly distributed in $\mathbf{Z}$.*

Proof. We proceed by Lemma 2. In order to prove that $([x_n])$ is uniformly distributed in $\mathbf{Z}$, we note that by Lemma 1 the sequence (x_n/m), $n=1, 2, \ldots$, is uniformly distributed mod 1 for any integer $m \geqq 2$. Therefore, $([x_n])$ is uniformly distributed in $\mathbf{Z}$ by a well-known theorem (see NIVEN [4] and KUIPERS and NIEDERREITER [2, Ch. 5, Theorem 1.4]).

Now let α be an irrational number. For any $(h_1, h_2) \in \mathbf{Z}^2$ with $(h_1, h_2) \neq (0, 0)$, the number $h_1\alpha + h_2$ is nonzero; therefore, the sequence $\big((h_1\alpha + h_2)x_n\big)$, $n=1, 2, \ldots$, is uniformly distributed mod 1 by Lemma 1. Hence, by [2, Ch. 1, Theorem 6.3], the sequence $\big((x_n\alpha, x_n)\big)$, $n=1, 2, \ldots$, in $\mathbf{R}^2$ is uniformly distributed mod 1 in $\mathbf{R}^2$. Now, for any sequence $\big((y_n, z_n)\big)$, $n=1, 2, \ldots$, in $\mathbf{R}^2$ which is uniformly distributed mod 1 in $\mathbf{R}^2$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{y_n\}, \{z_n\}) = \int_0^1 \int_0^1 f(y, z)\, dy\, dz \tag{2}$$

for any complex-valued continuous function f on $[0, 1]^2$ (see [2, Ch. 1, Theorem 6.1]), where $\{t\}$ denotes the fractional part of $t \in \mathbf{R}$.

For typographic convenience, we write $\exp(t) = e^{2\pi i t}$ for $t \in \mathbf{R}$. We choose a nonzero integer h. Then

$$\exp(h[x_n]\alpha) = \exp(hx_n\alpha - h\alpha\{x_n\}) = \exp(h\{x_n\alpha\} - h\alpha\{x_n\})$$

for all $n \geqq 1$. Hence, if we apply (2) to the sequence $\big((x_n\alpha, x_n)\big)$ and to the function $f(y, z) = \exp(hy - h\alpha z)$, we obtain

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \exp(h[x_n]\alpha) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \exp(h\{x_n\alpha\} - h\alpha\{x_n\})$$
$$= \int_0^1 \int_0^1 \exp(hy - h\alpha z)\, dy\, dz = 0.$$

This means that the sequence $([x_n]\alpha)$ $(n=1, 2, \ldots)$ is uniformly distributed mod 1. Since α was an arbitrary irrational number, the proof of the theorem is complete by Lemma 2.

Theorem 1 contains a variety of interesting special cases. We list a few of them.

Corollary 1. *Let $P(x)=\alpha_s x^s+\dots+\alpha_0$ be a polynomial over* **R** *of degree at least* 2. *If the system $\{\alpha_s, \alpha_{s-1}, \dots, \alpha_1\}$ has rank at least* 2 *over the rationals, then the sequence $([P(n)])$ $(n=1, 2, \dots)$ is Hartman-uniformly distributed in* **Z**.

Proof. This follows from [2, Ch. 4, Example 5.4] and Theorem 1.

We remark that VEECH [5] has even shown a somewhat stronger property of the sequence $([P(n)])$. Obviously, the sequence $(a_n)=([P(n)])$ is eventually increasing or eventually decreasing with $\lim_{n\to\infty} |a_{n+1}-a_n|=\infty$, and therefore

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{card}\left(E_N \cap (E_N+k)\right) = 0 \quad \text{for all} \quad k \geqq 1,$$

so that (1) fails drastically.

For positive integers r, we define the action of the difference operator Δ^r on sequences (x_n) in **R** recursively: we set $\Delta^1 x_n = x_{n+1}-x_n$ for $n\geqq 1$ and $\Delta^r x_n = \Delta^{r-1}(\Delta^1 x_n)$ for $r\geqq 2$ and $n\geqq 1$.

Corollary 2. *Let (x_n) be a sequence in* **R** *such that for some positive integer r the following properties are satisfied: $\Delta^r x_n$ tends monotonically to* 0 *as $n\to\infty$ and $\lim_{n\to\infty} n|\Delta^r x_n|=\infty$. Then the sequence $([x_n])$ is Hartman-uniformly distributed in* **Z**.

Proof. We note that for any nonzero $\alpha\in\mathbf{R}$ the sequence $(x_n\alpha)$ satisfies the same properties as (x_n). Therefore, by [2, Ch.1, Theorem 3.4], the sequence $(x_n\alpha)$ is uniformly distributed mod 1. The desired result follows from Lemma 1 and Theorem 1.

The following condition is usually easier to check.

Corollary 3. *Suppose the function $f(t)$ is defined for $t\geqq 1$ and r times differentiable for sufficiently large t and for some positive integer r. Furthermore, assume that $f^{(r)}(t)$ tends monotonically to* 0 *as $t\to\infty$ and that $\lim_{t\to\infty} t|f^{(r)}(t)|=\infty$. Then the sequence $([f(n)])$, $n=1, 2, \dots$, is Hartman-uniformly distributed in* **Z**.

Proof. One uses [2, Ch. 1, Theorem 3.5] and proceeds as in the proof of Corollary 2.

If we choose $\sigma>1$, $\sigma\notin\mathbf{Z}$, then the sequence $(a_n)=([n^\sigma])$, $n=1, 2, \dots$, is Hartman-uniformly distributed in **Z** by Corollary 3. On the other hand, the sequence (a_n) is increasing with $\lim_{n\to\infty}(a_{n+1}-a_n)=\infty$, so that we have again

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{card}\left(E_N \cap (E_N+k)\right) = 0 \quad \text{for all} \quad k \geqq 1.$$

In the following theorem, we go beyond the examples constructed above and show that for a Hartman-uniformly distributed sequence in $\mathbf{Z}$ the limits in (1) may have any prescribed value.

Theorem 2. *For a given positive integer k and a real number α with $0 \leqq \alpha \leqq 1$, there exists a Hartman-uniformly distributed sequence in* $\mathbf{Z}$ *with*

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{card}\left(E_N \cap (E_N + k)\right) = \alpha. \tag{3}$$

Proof. We have already constructed examples for $\alpha = 0$. If $\alpha = 1$, we may take the sequence of positive integers as an example. Now suppose $0 < \alpha < 1$, and let $m \geqq 1$ be an integer with $\alpha \leqq \frac{m}{m+1}$. By choosing a sequence $(a_n) = ([n^\sigma])$ $(n = 1, 2, \ldots)$ with a sufficiently large $\sigma \notin \mathbf{Z}$, we get a Hartman-uniformly distributed sequence in $\mathbf{Z}$ with $a_{n+1} - a_n > k(m+1)$ for all $n \geqq 1$. We set $\beta = \frac{m}{\alpha} - m$, so that $\beta \geqq 1$. Furthermore, we put $s(j) = [\beta j]$ for $j = 0, 1, \ldots$. Consider the following sequence:

$$\begin{gathered} a_1, a_2, \ldots, a_{s(1)}, a_1 + k, a_1 + 2k, \ldots, a_1 + mk, a_{s(1)+1}, a_{s(1)+2}, \ldots, \\ a_{s(2)}, a_2 + k, a_2 + 2k, \ldots, a_2 + mk, \ldots, a_{s(j-1)+1}, a_{s(j-1)+2}, \ldots, \\ a_{s(j)}, a_j + k, a_j + 2k, \ldots, a_j + mk, \ldots. \end{gathered}$$

Let us denote this sequence by (b_n). We show first that (b_n) is Hartman-uniformly distributed in $\mathbf{Z}$. Let χ be a nontrivial character of $\mathbf{Z}$. For $N > s(1) + m$, there exists a unique $j \geqq 1$ such that $s(j) + jm < N \leqq s(j+1) + (j+1)m$. Then

$$\left| \sum_{n=1}^{N} \chi(b_n) \right| \leqq \left| \sum_{n=1}^{s(j)+jm} \chi(b_n) \right| + s(j+1) - s(j) + m,$$

so that

$$\left| \frac{1}{N} \sum_{n=1}^{N} \chi(b_n) \right| \leqq \left| \frac{1}{s(j)+jm} \sum_{n=1}^{s(j)+jm} \chi(b_n) \right| + \frac{s(j+1) - s(j) + m}{s(j)+jm}.$$

Therefore, it suffices to show that

$$\lim_{j\to\infty} \frac{1}{s(j)+jm} \sum_{n=1}^{s(j)+jm} \chi(b_n) = 0. \tag{4}$$

For $j \geqq 1$ we have

$$\sum_{n=1}^{s(j)+jm} \chi(b_n) = \sum_{n=1}^{s(j)} \chi(a_n) + \sum_{p=1}^{m} \sum_{q=1}^{j} \chi(a_q + pk),$$

and therefore

$$\left| \frac{1}{s(j)+jm} \sum_{n=1}^{s(j)+jm} \chi(b_n) \right| \leqq \left| \frac{1}{s(j)} \sum_{n=1}^{s(j)} \chi(a_n) \right| + \left| \frac{1}{j} \sum_{q=1}^{j} \chi(a_q) \right|. \tag{5}$$

But since (a_n) is Hartman-uniformly distributed in $\mathbf{Z}$, the right-hand side of (5) tends to 0 as $j\to\infty$, and so (4) is established.

It remains to prove that (b_n) satisfies (3). For $N>s(1)+m$, there is a unique $j\geqq 1$ with $s(j)+jm<N\leqq s(j+1)+(j+1)m$. Then

$$\frac{s(j)+jm}{s(j+1)+(j+1)m}\cdot\frac{\operatorname{card}\left(E_{s(j)+jm}\cap(E_{s(j)+jm}+k)\right)}{s(j)+jm}\leqq\frac{\operatorname{card}\left(E_N\cap(E_N+k)\right)}{N}\leqq$$

$$\leqq\frac{\operatorname{card}\left(E_{s(j+1)+(j+1)m}\cap(E_{s(j+1)+(j+1)m}+k)\right)}{s(j+1)+(j+1)m}\cdot\frac{s(j+1)+(j+1)m}{s(j)+jm},$$

nd since

$$\lim_{j\to\infty}\frac{s(j+1)+(j+1)m}{s(j)+jm}=1,$$

it suffices to show that

$$\lim_{j\to\infty}\frac{\operatorname{card}\left(E_{s(j)+jm}\cap(E_{s(j)+jm}+k)\right)}{s(j)+jm}=\alpha. \tag{6}$$

Using $a_{n+1}-a_n>k(m+1)$ for all $n\geqq 1$, it follows easily that for every $j\geqq 1$ we have

$$E_{s(j)+jm}\cap(E_{s(j)+jm}+k)=\bigcup_{q=1}^{j}\{a_q+k,\,a_q+2k,\,\ldots,\,a_q+mk\}.$$

We conclude that

$$\lim_{j\to\infty}\frac{\operatorname{card}\left(E_{s(j)+jm}\cap(E_{s(j)+jm}+k)\right)}{s(j)+jm}=\lim_{j\to\infty}\frac{jm}{s(j)+jm}=\frac{m}{\beta+m}=\alpha,$$

and so (6) is shown. This completes the proof of Theorem 2.

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(Received January 11, 1974)