

## Dissipative $J$ -self-adjoint operators and associated $J$ -isometries

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**1. Introduction.** Only bounded operators on a Hilbert space  $\mathfrak{H}$  will be considered in this paper. Let  $J$  be self-adjoint. If  $A$  is any operator satisfying

$$(1.1) \quad JA = A^*J,$$

then  $A$  will be called  $J$ -self-adjoint; similarly, if  $V$  satisfies

$$(1.2) \quad V^*JV = J,$$

$V$  will be called  $J$ -isometric or a  $J$ -isometry. This terminology corresponds to that in the literature dealing with geometry of spaces having indefinite metrics. Thus, if  $(x, y)$  is the usual inner product on  $\mathfrak{H}$  and if one introduces the modified inner product  $(x, y)_J = (Jx, y)$  then  $A$  is  $J$ -self-adjoint if  $(Ax, y)_J = (x, Ay)_J$  for all  $x, y$  in  $\mathfrak{H}$ . This is the same as  $(JAx, y) = (Jx, Ay)$ , that is, (1.1). Similarly,  $V$  is  $J$ -isometric if  $(Vx, Vy)_J = (x, y)_J$  for all  $x, y$  in  $\mathfrak{H}$ , which is equivalent to (1.2). See, in particular, the surveys by KREIN [5] and NAIMARK and ISMAGILOV [6] where, for the most part, it is assumed that  $J^2 = I$ . Another kind of indefinite scalar product is considered by BEREZIN [1]. In the present paper, the aforementioned restriction  $J^2 = I$  will be considerably relaxed (see (1.5) below) but additional conditions ((1.3), (1.4)) will be imposed on the operators  $A$  and  $V$  of (1.1) and (1.2).

Throughout it will be supposed that if  $A$  satisfies (1.1) then  $\text{Im } (A) = (A - A^*)/2i$  satisfies

$$(1.3) \quad \text{either } \text{Im } (A) \geq 0 \quad \text{or} \quad \text{Im } (A) \leq 0.$$

An operator  $A$  will be called dissipative if the first part of (1.3) holds; thus, condition (1.3) is that either  $A$  or  $-A$  be dissipative. (This definition coincides with that of SZ.-NAGY and FOIAŞ [9], p. 167. It should be noted, however, that sometimes  $A$

is said to be dissipative if  $\operatorname{Re}(A) \leq 0$ ; see, e.g., KATO [4], p. 279). Further, it will be supposed that if  $V$  satisfies (1.2) then

$$(1.4) \quad 1 \notin \operatorname{sp}(V) \quad \text{and either} \quad VV^* \leq I \quad \text{or} \quad VV^* \equiv I.$$

The first inequality of (1.4) is of course equivalent to  $\|V\| \leq 1$ , that is, that  $V$  is a contraction. Incidentally, if (1.2) holds then  $\|J\| \leq \|J\| \|V\|^2$  so that, unless  $J=0$ , necessarily  $\|V\| \geq 1$ .

It will be convenient to recall the notion of the absolutely continuous part of a self-adjoint operator  $J$ . If  $J$  has the spectral resolution  $J = \int t dE_t$ , then the set,  $\mathfrak{H}_a(J)$ , of vectors  $x$  in  $\mathfrak{H}$  for which  $\|E_t x\|^2$  is an absolutely continuous function of  $t$  is a subspace of  $\mathfrak{H}$  invariant under  $J$ . If  $\mathfrak{H}_a(J) \neq 0$ , the restriction  $J_a = J|_{\mathfrak{H}_a(J)}$  is called the absolutely continuous part of  $J$ ; in particular,  $J$  is said to be absolutely continuous if  $J = J_a$ . (See, e.g., HALMOS [3], p. 104, KATO [5], p. 516.) For later use, let  $P_0(J) = \{x: Jx=0\}$ ; clearly,  $\mathfrak{H}_a(J) \perp P_0(J)$ .

**Theorem 1.** *Let  $J$  be self-adjoint and suppose that  $J$  and  $A$  are bounded operators on a Hilbert space  $\mathfrak{H}$  satisfying (1.1), (1.3) and*

$$(1.5) \quad J \neq J_a \oplus 0, \quad \text{that is,} \quad \mathfrak{H}_a(J) \oplus P_0(J) \quad \text{is a proper subspace of } \mathfrak{H}.$$

*Then there exists a subspace  $\mathfrak{M}$  satisfying*

$$(1.6) \quad \mathfrak{M} \supset (\mathfrak{H}_a(J) \oplus P_0(J))^\perp \neq 0,$$

*reducing both  $A$  and  $J$  and for which*

$$(1.7) \quad A|_{\mathfrak{M}} \quad \text{is self-adjoint.}$$

It is understood that either term in the direct sum on the right side of the inequality (1.5) may be absent, that is, that either  $\mathfrak{H}_a(J)$  or  $P_0(J)$  may be the 0 space. In particular, if  $J$  has no absolutely continuous part and if 0 is not in the point spectrum of  $J$  then  $\mathfrak{M}$  of (1.6) is  $\mathfrak{H}$  and so, by (1.7),  $A$  is self-adjoint.

**Theorem 2.** *Let  $J$  be self-adjoint and suppose that  $J$  and  $V$  are bounded operators on a Hilbert space  $\mathfrak{H}$  satisfying (1.2), (1.4) and (1.5). Then there exists a subspace  $\mathfrak{M}$  satisfying (1.6), reducing both  $V$  and  $J$  and for which*

$$(1.8) \quad V|_{\mathfrak{M}} \quad \text{is unitary.}$$

The proof of Theorem 1 will be given in section 2 and will depend on a general result on commutators in PUTNAM [7], p. 20. The proof of Theorem 2 will be derived in section 3 as a corollary of Theorem 1 via the Cayley transform. Some remarks on the Theorems as well as some applications will be given in section 4.

## 2. Proof of Theorem 1. In view of (1.1),

$$(2.1) \quad AJ - JA = (A - A^*)J,$$

therefore,

$$(2.2) \quad (JA)J - J(JA) = iC, \quad \text{where } C = 2J(\text{Im}(A))J.$$

Let  $\mathfrak{N}$  denote the least subspace of  $\mathfrak{H}$  reducing both self-adjoint operators  $JA$  and  $J$  and containing the range of the self-adjoint operator  $C$ . By (1.3), either  $C \geq 0$  or  $C \leq 0$ , and so, by the Theorem of [7], p. 20,  $\mathfrak{N} \subset (\mathfrak{H}_a(J) \cap \mathfrak{H}_a(JA)) \subset \mathfrak{H}_a(J)$ , hence  $\mathfrak{N}^\perp \supset (\mathfrak{H}_a(J))^\perp$ . In addition, it is clear that  $\mathfrak{N}^\perp$  reduces both  $J$  and  $JA$  (and  $C$ ) and that  $C|_{\mathfrak{N}^\perp} = 0$ . Thus, if  $x \in \mathfrak{N}$ ,  $0 = (Cx, x) = 2(\text{Im}(A)Jx, Jx)$ , hence, since  $\text{Im}(A)$  is semi-definite,

$$(2.3) \quad \text{Im}(A)Jx = 0 \quad \text{for } x \in \mathfrak{N}^\perp.$$

Next, note that  $P_0(J) \subset (\mathfrak{H}_a(J))^\perp \subset \mathfrak{N}^\perp$  and that  $\mathfrak{N}^\perp \ominus P_0(J) \supset (\mathfrak{H}_a(J) \oplus P_0(J))^\perp \neq 0$ , the last inequality by (1.5). Let

$$(2.4) \quad \mathfrak{M} = \mathfrak{N}^\perp \ominus P_0(J) \quad (\neq 0).$$

It is clear that  $\mathfrak{M}$  reduces  $J$ . Also, if  $x \in \mathfrak{M}$  and  $y \in P_0(J)$  then  $(JAx, y) = (Ax, Jy) = 0$ , so that, since  $\mathfrak{N}$  reduces  $JA$ , so also does  $\mathfrak{M}$ . Thus,

$$(2.5) \quad \mathfrak{M} \text{ reduces } J \text{ and } JA.$$

Further,

$$(2.6) \quad J(\mathfrak{M}) \text{ is dense in } \mathfrak{M}.$$

In fact, otherwise, there would exist a vector  $y \in \mathfrak{M}$ ,  $y \neq 0$ , such that  $0 = (Jx, y) = (x, Jy)$  for all  $x \in \mathfrak{M}$ . Hence  $y \in P_0(J)$  and hence  $y \in M \cap P_0(J)$ , so  $y = 0$ , a contradiction.

It now follows from (2.1), (2.3) and (2.6) that

$$(2.7) \quad AJx = JAx \quad \text{for } x \in \mathfrak{M}.$$

In view of (2.5) and (2.6), this implies that  $\mathfrak{M}$  is invariant under  $A$ . Finally, relations (1.1), (2.5) and (2.6) imply that  $\mathfrak{M}$  is also invariant under  $A^*$ . Thus,  $\mathfrak{M}$  reduces  $A$  and relations (2.1), (2.6) and (2.7) imply (1.7).

**3. Proof of Theorem 2.** Since  $1 \notin \text{sp}(V)$ , the operator  $A = i(I + V)(I - V)^{-1}$  is bounded. Further it is easily verified that  $-i \notin \text{sp}(A)$  and that  $V$  is the Cayley transform of  $A$ , that is

$$(3.1) \quad V = (A - iI)(A + iI)^{-1} \quad \text{and} \quad A = i(I + V)(I - V)^{-1}.$$

A straightforward calculation shows that  $A$  satisfies (1.1) if and only if  $V$  satisfies (1.2). Furthermore,  $(I-V)(\operatorname{Im}(A))(I-V^*)=I-VV^*$ , so that  $\operatorname{Im}(A) \geq 0$  or  $\leq 0$  according as  $I-VV^* \geq 0$  or  $\leq 0$ ; in this connection, see [9], p. 357.

In order to prove Theorem 2 one need only define  $A$  as in (3.1) and then apply Theorem 1 to  $A$ . Then the space  $\mathfrak{M}$  of Theorem 1 clearly reduces  $V$  while (1.7) implies (1.8) by the well-known properties of the Cayley transform.

**4. Remarks.** It may be noted that the first part of (1.4), namely, that 1 not be in the spectrum of  $V$ , is essential in Theorem 2 for the validity of assertion (1.8). In fact, if  $J=I$  and if  $V$  denotes the unilateral shift, then, although  $1 \in \operatorname{sp}(V)$  (in fact,  $\operatorname{sp}(V)$  is the unit disk  $\{z: |z| \leq 1\}$ ), nevertheless,  $V$  is a contraction,  $V$  and  $J$  satisfy (1.2) and (1.5), and  $V$  is irreducible, so that, in particular,  $V$  has no unitary part; cf. [3], p. 73.

It is clear that if (1.1) holds and if  $J$  is non-singular, then  $A^*$  is similar to  $A$  and hence  $A$  and  $A^*$  have identical spectra. Further, condition (1.3) implies that the spectrum of  $A$  lies either in the upper half-plane or in the lower half-plane. Thus, if  $J$  is non-singular then (1.1) and (1.3) imply that the spectrum of  $A$  is real. Hence, for instance, if  $A$  is also normal it is necessarily self-adjoint. On the other hand, there exist non-singular self-adjoint operators  $J$  and dissipative operators  $A$  for which (1.1) holds and for which  $A$  is completely non-self-adjoint, that is,  $A$  has no reducing space on which it is self-adjoint. It follows from Theorem 1 that such an operator  $J$  is necessarily absolutely continuous.

To obtain such a pair  $J$  and  $A$ , let  $A$  be the operator on  $\mathfrak{H}=L^2(0, 1)$  defined by

$$(Ax)(t) = t x(t) + i \int_0^t x(s) ds.$$

Then  $A$  is dissipative (see [9], p. 365). In addition,  $A$  is completely non-self-adjoint and is similar to the self-adjoint multiplication operator  $A_0=t$  on  $L^2(0, 1)$ . (This result is due to SAHNOVIČ; see [9], pp. 368, 372.) Let  $T$  denote any non-singular operator  $T$  for which  $A=TA_0T^{-1}$ . If  $T$  has the polar factorization  $T=PU$  where  $P$  is positive and  $U$  is unitary, then  $A=PUA_0U^*P^{-1}$  and  $A^*=P^{-1}UA_0U^*P=P^{-2}AP^2$ , so that (1.1) holds with  $J=P^{-2}$ . It follows from Theorem 1 that  $P^{-2}$ , hence also  $P$ , must be absolutely continuous.

It is clear from the above argument that if  $T$  is non-singular with the polar factorization  $T=PU$  and if  $B$  is any self-adjoint operator then (1.1) holds with  $A=TBT^{-1}$  and  $J=P^{-2}$ .

Concerning not necessarily bounded dissipative operators and, in particular, ones similar to self-adjoint operators, see SZ.-NAGY and FOIAŞ [9], Chapt. IX, §§ 4, 5, as well as their paper [8].

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