Subnormal limits of nilpotent operators

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1. Introduction. Let $\mathfrak{H}$ be a fixed separable, infinite dimensional complex Hilbert space and let $\mathcal{L}(\mathfrak{H})$ denote the algebra of all (bounded, linear) operators on $\mathfrak{H}$. In [3, Problem 7] HALMOS has asked for a characterization of the set of all operators in $\mathcal{L}(\mathfrak{H})$ which are uniform limits of nilpotent operators. In what follows we shall denote by $N(\mathfrak{H})$ the set of all nilpotent operators on $\mathfrak{H}$. In the recent paper [5] HERRERO made a remarkable contribution to Halmos' problem by showing that a normal operator is in the uniform closure $\overline{N(\mathfrak{H})}$ of $N(\mathfrak{H})$ if and only if its spectrum is connected and contains the origin [5, Theorem 7]. In his paper Herrero asked whether the direct sum of a unilateral shift in $\mathcal{L}(\mathfrak{H})$ and a normal operator on $\mathfrak{H}$ whose spectrum coincides with the closed unit disk is in $\overline{N(\mathfrak{H}\oplus\mathfrak{H})}$. In the present note we answer this question in the affirmative. Actually, we prove a more general result making a further progress in the solution of Halmos' question.

Our main theorem can be stated as follows:

Theorem 1.1. *If T is a subnormal operator on $\mathcal{L}(\mathfrak{H})$ whose approximate point spectrum is simply connected and contains the origin, then T is the uniform limit of nilpotent operators.*

The proof of the above theorem requires some auxiliary results and will be given in Section 2. In the final section of this paper we introduce a subclass of $\overline{N(\mathfrak{H})}$ which we call the class of pseudonilpotent operators and we study some of its properties. The main characteristic of these operators is that its nilpotent approximants are easy to determine. From this point of view, pseudonilpotent operators are perhaps more tractable than an arbitrary operator in $\overline{N(\mathfrak{H})}$.

2. Proof of Theorem 1.1. Throughout, for a given operator T in $\mathcal{L}(\mathfrak{H})$ we shall denote by $\sigma(T)$ the spectrum of T and by $\sigma_l(T)$ the left spectrum of T (or approximate point spectrum of T). Furthermore, we denote by $E(T)$ the essential spectrum of T and by $E_l(T)$ the left essential spectrum of T. (We recall that $E(T)$ and $E_l(T)$ are the spectrum and the left spectrum, respectively of the image of T in the Calkin algebra.) Also, in what follows D will denote the closed unit disk of the complex plane.

Theorem 2.1. *Let T be a subnormal operator in $\mathscr{L}(\mathfrak{H})$ such that $\sigma(T)\subset D$ and let M be a normal operator in $\mathscr{L}(\mathfrak{H})$ such that $\sigma(M)=D$. Then $T\oplus M\in\overline{N(\mathfrak{H}\oplus\mathfrak{H})}$.*

Proof. If T is a normal operator, then $T\oplus M$ is a normal operator whose spectrum is connected and contains the origin and hence the theorem follows from [5, Theorem 7]. Therefore, we may assume that T is not normal. Let $\mathfrak{K}$ be a complex Hilbert space and let N be a normal operator in $\mathscr{L}(\mathfrak{K})$ such that N is a minimal normal extension of the subnormal operator T, i.e. $\mathfrak{H}$ is an invariant subspace of N such that $N|\mathfrak{H}=T$ and the smaller reducing subspace of N containing $\mathfrak{H}$ coincides with $\mathfrak{K}$ [2]. Since T is not normal it is easy to see that $\mathfrak{K}\ominus\mathfrak{H}$ is infinite dimensional. Thus, after an identification via a suitable unitary transformation, we can assume that $\mathfrak{K}=\mathfrak{H}\oplus\mathfrak{H}$ and that N can be represented by the 2×2 operator matrix

$$N=\begin{bmatrix} S & O_{\mathfrak{H}} \\ R & T \end{bmatrix},$$

where R and S are in $\mathscr{L}(\mathfrak{H})$ and $0_{\mathfrak{H}}$ is the zero operator on $\mathfrak{H}$. It also follows that $\sigma(N)\subset\sigma(T)$ and hence $\sigma(N)\subset\sigma(M)$. Now we observe that for every $n=1, 2, \ldots$

$$D=\sigma(M)=\sigma\left\{\left[\sum_{j=1}^{n}\oplus\frac{n-j}{n}(M\oplus N)\right]\oplus M\right\}.$$

An easy exercise in spectral theory shows that for every $\varepsilon>0$ and every $n=1, 2, \ldots$ there exists a unitary transformation $U_{\varepsilon,n}\colon\mathfrak{H}\to\sum_{1}^{3n+1}\oplus\mathfrak{H}$ such that

$$\left\|U_{\varepsilon,n}MU_{\varepsilon,n}^{-1}-\left\{\left[\sum_{j=1}^{n}\oplus\frac{n-j}{n}(M\oplus N)\right]\oplus M\right\}\right\|<\varepsilon.$$

Therefore, since $M\in\overline{N(\mathfrak{H})}$ (cf. [5, Theorem 7]), it follows that in order to prove that $T\oplus M\in\overline{N(\mathfrak{H}\oplus\mathfrak{H})}$ it suffices to establish the following assertion: For every $n=2, 3, \ldots$ there exists and operator in $N\left(\sum_{1}^{3n+1}\oplus\mathfrak{H}\right)$ whose distance to the operator

$$A_n=T\oplus\left[\sum_{j=1}^{n}\oplus\frac{n-j}{n}(M\oplus N)\right]$$

is less that $\frac{1}{n}$. To prove this fact, for $n=2, 3, \ldots$, let

$$B_n=\frac{n-1}{n}T\oplus\left\{\sum_{j=1}^{n-1}\oplus\left[\frac{n-j}{n}(M\oplus N)-\frac{1}{n}(0_{\mathfrak{H}}\oplus 0_{\mathfrak{H}}\oplus T)\right]\right\}\oplus 0_{\mathfrak{H}}\oplus 0_{\mathfrak{H}}\oplus 0_{\mathfrak{H}}.$$

Since $A_n-B_n=\left[\sum_{j=1}^{n}\oplus\frac{1}{n}(T\oplus 0_{\mathfrak{H}}\oplus 0_{\mathfrak{H}})\right]\oplus 0_{\mathfrak{H}}$ we deduce that $\|A_n-B_n\|\leq\frac{1}{n}$, $n=$ $=2, 3, \ldots$. Now we show that the operator B_n is the uniform limit of nilpotent operators. In fact, the operator B_n can be represented as the direct sum of a lower triangular $n\times n$ matrix whose j-th diagonal term is $\frac{n-j}{n}(T\oplus M\oplus S)$ and the operator

$0_{\mathfrak{H}}$. Thus, in order to complete the proof of theorem (cf. [5, Theorem 5]) it remains to show that $T\oplus M\oplus S\in\overline{N(\mathfrak{H}\oplus\mathfrak{H}\oplus\mathfrak{H})}$. To see this let's observe first that

$$S\oplus T=\lim_{\varepsilon\to 0}\begin{bmatrix} S & 0_{\mathfrak{H}} \\ \varepsilon R & T\end{bmatrix}=\lim_{\varepsilon\to 0}\begin{bmatrix} 1_{\mathfrak{H}} & 0_{\mathfrak{H}} \\ 0_{\mathfrak{H}} & \varepsilon 1_{\mathfrak{H}}\end{bmatrix}\begin{bmatrix} S & 0_{\mathfrak{H}} \\ R & T\end{bmatrix}\begin{bmatrix} 1_{\mathfrak{H}} & 0_{\mathfrak{H}} \\ 0_{\mathfrak{H}} & \frac{1}{\varepsilon} 1_{\mathfrak{H}}\end{bmatrix}.$$

In view of the last remark, the fact that $\overline{N(\mathfrak{H}\oplus\mathfrak{H}\oplus\mathfrak{H})}$ is invariant under similarities and since

$$N\oplus M=\begin{bmatrix} S & 0_{\mathfrak{H}} \\ R & T\end{bmatrix}\oplus M$$

is in $\overline{N(\mathfrak{H}\oplus\mathfrak{H}\oplus\mathfrak{H})}$ we conclude that $S\oplus T\oplus M$ and hence $T\oplus M\oplus S$ is in $\overline{N(\mathfrak{H}\oplus\mathfrak{H}\oplus\mathfrak{H})}$, as desired.

Corollary 2.2. *Let U be a unilateral shift in $\mathscr{L}(\mathfrak{H})$ and let M be a normal operator in $\mathscr{L}(\mathfrak{H})$ such that $\sigma(M)=D$. Then $U\oplus M\in\overline{N(\mathfrak{H}\oplus\mathfrak{H})}$.*

The following lemma generalizes a result in [5].

Lemma 2.3. *Let $T\in\overline{N(\mathfrak{H})}$ and let S be an operator in the uniformly closed, inverse closed algebra generated by T. Then $TS\in\overline{N(\mathfrak{H})}$.*

Proof. By hypothesis there exists a sequence $\{f_n\}$ of rational functions with poles off $\sigma(T)$ such that $\lim_{n\to\infty}\|S-f_n(T)\|=0$. Also, there exists a sequence $\{Q_k\}$ in $N(\mathfrak{H})$ such that $\lim_{k\to\infty}\|T-Q_k\|=0$. Now let k_1 be the first positive integer such that $\|f_1(T)-f_1(Q_{k_1})\|<1$; having defined k_n, $n\geqq 1$ let k_{n+1} be the first positive integer greater than k_n such that $\|f_{n+1}(T)-f_{n+1}(Q_{k_{n+1}})\|<\frac{1}{n+1}$. Letting $R_n=Q_{k_n}$ $(n=1, 2, \ldots)$ it readily follows that $\lim_{n\to\infty}\|TS-R_nf_n(R_n)\|=0$. Since $R_nf_n(R_n)\in N(\mathfrak{H})$ $(n=1, 2, \ldots)$ we conclude that $TS\in\overline{N(\mathfrak{H})}$.

Proof of theorem 1.1. Let T be a subnormal operator in $\mathscr{L}(\mathfrak{H})$ such that $\sigma_l(T)$ is simply connected and contains the origin. It follows that $\sigma(T)=\sigma_l(T)-E(T)=E_l(T)$. From [6] we deduce that, up to a small norm perturbation, T is unitarily equivalent to an operator of the form $T\oplus T'$ where T' is any normal operator in $\mathscr{L}(\mathfrak{H})$ such that $\sigma(T')=E(T')=\sigma(T)$. Since, up to a small norm perturbation and a unitary equivalence, T' can be replaced in the above direct sum by a normal operator N in $\mathscr{L}(\mathfrak{H})$ whose spectrum $\sigma(N)$ is "closed" to $\sigma(T)$ in the Hausdorff metric topology, and such that $\sigma(N)$ is simply connected, has smooth boundary and contains $\sigma(T)$ in its interior, in order to complete the proof of the theorem it suffices to prove that $T\oplus N\in\overline{N(\mathfrak{H}\oplus\mathfrak{H})}$, for any normal operator N in $\mathfrak{H}$ satisfying the properties described previously. Let φ be a homeomorphism from $\sigma(N)$ onto D such that $\varphi(0)=0$, φ is analytic in the interior of $\sigma(N)$ and φ maps the boundary of $\sigma(N)$ onto the boundary of D. (The existence of this function φ can be deduced from standard facts in the theory of conformal mappings.) Since $\sigma(N)$ is simply connected and φ is analytic on $\sigma(T)$ it follows that $\varphi(T)$ is subnormal and $\sigma[\varphi(T)]=\varphi[\sigma(T)]$ is contained in the interior of D. Employing the fact that $\varphi(N)$ is normal and $\sigma[\varphi(N)]=D$, from Theorem 2.1 we deduce that $\varphi(T)\oplus\varphi(N)\in\overline{N(\mathfrak{H}\oplus\mathfrak{H})}$. Let $\psi\colon D\to\sigma(N)$ be the inverse function of φ. Since $\psi(0)=0$, there exists a continuous complex valued function η on D which is analytic on the interior of D and satisfies $\psi(\lambda)=\lambda\eta(\lambda)$, for every $\lambda\in D$. Observing that $\eta[\varphi(T)]\oplus\eta[\varphi(N)]$ is in the uniformly closed, inverse closed algebra generated by $\varphi(T)\oplus\varphi(N)$ and using Lemma 2.3 we conclude that

$$T\oplus N(=\psi[\varphi(T)]\oplus\psi[\varphi(N)]=\varphi(T)\eta[\varphi(T)]\oplus\varphi(N)\eta[\varphi(N)]=$$
$$=[\varphi(T)\oplus\varphi(N)]\{\eta[\varphi(T)]\oplus\eta[\varphi(N)]\})$$

is in $\overline{N(\mathfrak{H}\oplus\mathfrak{H})}$, as asserted.

Remark 2.4. Since the subset of elements with connected spectrum in any complex Banach algebras with identity is closed in the norm topology (cf. [8, Theorem 3]) and the spectrum is an uppersemicontinuous function, it follows that every operator T in $\overline{N(\mathfrak{H})}$ must satisfy

(*) $\sigma(T)$ and $E(T)$ are connected and $E(T)$ contains the origin.

Let $\Omega(T)=\{\lambda\in\sigma(T)\colon T-\lambda$ is not a Fredholm operator of index zero$\}$. From the continuity properties of the index function on the set of semi-Fredholm operators in $\mathscr{L}(\mathfrak{H})$ we see that every operator T in $\overline{N(\mathfrak{H})}$ must also satisfy

(**) $$\Omega(T)=E_l(T)\cap E_r(T).$$

(Recall that $E_r(T)$ is the conjugate set of $E_l(T^*)$). Conversely, we conjecture that if an operator T in $\mathscr{L}(\mathfrak{H})$ satisfies conditions (*) and (**), then $T\in\overline{N(\mathfrak{H})}$. The validity of this conjecture would imply of course that every quasinilpotent operator on $\mathfrak{H}$ is in $\overline{N(\mathfrak{H})}$, answering in the affirmative Problem 7 of [3].

Let $QT(\mathfrak{H})$ be the set of all quasitriangular operators on $\mathfrak{H}$, i.e. $T\in QT(\mathfrak{H})$ if and only if there exists an increasing sequence $\{P_n\}$ in $\mathscr{L}(\mathfrak{H})$ of finite rank projections tending strongly to the identity such that $\lim_{n\to\infty}\|TP_n-P_nTP_n\|=0$ (cf. [3, Problem 4]). From the spectral characterization of quasitriangular operators given in [1] it readily follows that $QT(\mathfrak{H})\cap QT(\mathfrak{H})^*=\{T\in\mathscr{L}(\mathfrak{H})\colon \Omega(T)=E_l(T)\cap E_r(T)\}$. Following the same circle of ideas of the above comments, we conjecture that $QT(\mathfrak{H})\cap QT(\mathfrak{H})^*$ is the uniform closure of the set of all algebraic operators on $\mathfrak{H}$. (We recall that an operator T in $\mathscr{L}(\mathfrak{H})$ is called an algebraic operator if there exists a polynomial p such that $p(T)=0$.) The results of [4], [5] and those of the present paper give partial affirmative answers to the above conjectures.

It may also be worth noting that if the first conjecture were true, it would follow from the above mentioned theorem of [1] that every operator in $\mathcal{L}(\mathfrak{H})$ which is not in $\overline{N(\mathfrak{H})}$ has a non-trivial invariant subspace, thereby reducing the invariant subspace problem to operators in $\overline{N(\mathfrak{H})}$.

3. Pseudonilpotent operators. In the rest of the paper it will be convenient to adopt the following terminology. Let $\mathfrak{H}_1, \dots, \mathfrak{H}_n$ be an orthogonal family of n subspaces of $\mathfrak{H}$ such that $\mathfrak{H} = \sum_{j=1}^{n} \oplus \mathfrak{H}_j$. Then every T in $\mathcal{L}(\mathfrak{H})$ can be represented, on the decomposition $\mathfrak{H} = \sum_{j=1}^{n} \oplus \mathfrak{H}_j$, by an $n \times n$ matrix of the form

$$T = \begin{bmatrix} T_{11} & T_{12} & \dots & T_{1n} \\ T_{21} & T_{22} & \dots & T_{2n} \\ \vdots & \vdots & & \vdots \\ T_{n1} & T_{n2} & \dots & T_{nn} \end{bmatrix}$$

where T_{ij} is a bounded, linear transformation from $\mathfrak{H}_j$ into $\mathfrak{H}_i$, $1 \leqq i, j \leqq n$. The operator in $\mathcal{L}(\mathfrak{H})$ represented by the lower triangular matrix

$$\begin{bmatrix} T_{11} & 0 & & \dots & 0 \\ T_{21} & T_{22} & 0 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ & & & & 0 \\ T_{n1} & T_{n2} & & \dots & T_{nn} \end{bmatrix}$$

will be called the lower triangular part of T with respect to the decomposition $\mathfrak{H} = \sum_{j=1}^{n} \oplus \mathfrak{H}_j$. Similarly, the upper triangular part of T with respect to the decomposition $\mathfrak{H} = \sum_{j=1}^{n} \oplus \mathfrak{H}_j$ is the operator in $\mathcal{L}(\mathfrak{H})$ represented by the upper triangular matrix

$$\begin{bmatrix} T_{11} & T_{12} & \dots & T_{1n} \\ 0 & T_{22} & \dots & T_{2n} \\ \vdots & & \ddots & \\ 0 & 0 & \dots & T_{nn} \end{bmatrix}.$$

Definition 3.1. Let $T \in \mathcal{L}(\mathfrak{H})$. We say that T is a *pseudonilpotent operator* if for every $\varepsilon > 0$ there exists a decomposition of $\mathfrak{H}$ into the direct sum of a finite orthogonal family of subspaces $\mathfrak{H}_1, \dots, \mathfrak{H}_n$ such that the norm of the lower triangular part of T with respect to the decomposition $\mathfrak{H} = \sum_{j=1}^{n} \oplus \mathfrak{H}_j$ is less than ε. The set of all pseudonilpotent operators in $\mathcal{L}(\mathfrak{H})$ will be denoted by $P(\mathfrak{H})$.

Remark 3.2. a) By interchanging the subspaces $\mathfrak{H}_1, \dots, \mathfrak{H}_n$ in definition 3.1 it is easy to see that an operator T is in $P(\mathfrak{H})$ if and only if for every $\varepsilon > 0$ there exists a decomposition of $\mathfrak{H}$ into the direct sum of a finite orthogonal family of subspaces such that the norm of the upper triangular part of T with respect to this decomposition is less than ε.

b) From a) it readily follows that $P(\mathfrak{H})=P(\mathfrak{H})^*$.

c) From definition 3.1 and the fact that if $T\in N(\mathfrak{H})$, then there always exists a decomposition $\mathfrak{H}=\sum_{j=1}^{n}\oplus\mathfrak{H}_j$ with respect to which the lower triangular part of T has norm zero, we see that the following inclusion formula holds:

$$N(\mathfrak{H})\subset P(\mathfrak{H})\subset\overline{N(\mathfrak{H})}.$$

In the following two theorems we shall see that these inclusions are actually proper.

Theorem 3.3. *Let A be a non-zero positive operator in $\mathscr{L}(\mathfrak{H})$ such that $\sigma(A)$ is connected. Then $A\in\overline{N(\mathfrak{H})}$, but $A\notin P(\mathfrak{H})$.*

Proof. From [5, Theorem 7] we deduce that $A\in\overline{N(\mathfrak{H})}$, thus it remains to show that $A\notin P(\mathfrak{H})$. Let $\mathfrak{H}_1, \ldots, \mathfrak{H}_n$ be an orthogonal family of subspaces of $\mathfrak{H}$ such that $\mathfrak{H}=\sum_{j=1}^{n}\oplus\mathfrak{H}_j$ and let A be represented by

$$A=\begin{bmatrix} A_{11} & A_{12} \ldots A_{1n} \\ A_{21} & A_{22} \ldots A_{2n} \\ \vdots & \ddots \vdots \\ A_{n1} & A_{n2} \ldots A_{nn} \end{bmatrix}$$

on the decomposition $\mathfrak{H}=\sum_{j=1}^{n}\oplus\mathfrak{H}_j$. Let B and C be the operators in $\mathscr{L}(\mathfrak{H})$ defined by

$$B=\begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \ldots 0 \\ \vdots & & \ddots \\ A_{n1} & A_{n2} & \ldots\ A_{nn} \end{bmatrix}, \quad C=\begin{bmatrix} A_{11} & 0 & \ldots & 0 \\ 0 & A_{22} & 0 \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & A_{nn} \end{bmatrix}.$$

It follows that $A+C=B+B^*$. Since C is a positive operator we infer that

$$\|A\| = \sup_{\substack{x\in\mathfrak{H}\\ \|x\|=1}} (Ax, x) \leqq \sup_{\substack{x\in\mathfrak{H}\\ \|x\|=1}} ([A+C]x, x) = \|A+C\| = \|B+B^*\| \leqq 2\|B\|,$$

and hence $\|B\|\geqq\|A\|/2$. Observing that B is the lower triangular part of A with respect to the decomposition $\mathfrak{H}=\sum_{j=1}^{n}\oplus\mathfrak{H}_j$, and since the family $\mathfrak{H}_1, \ldots, \mathfrak{H}_n$ is arbitrary we conclude that $A\notin P(\mathfrak{H})$.

Theorem 3.4. *If K is a quasinilpotent compact operator on $\mathfrak{H}$, then $K\in P(\mathfrak{H})$.*

Proof. Let K be any quasinilpotent compact operator in $\mathscr{L}(\mathfrak{H})$. Then there exists an increasing sequence $\{P_n\}$ in $\mathscr{L}(\mathfrak{H})$ of finite rank projections tending strongly to the identity such that $\lim_{n\to\infty} \|K-P_nKP_n\|=0$. From the upper semicontinuity of the spectrum, given $\varepsilon>0$ there exists a positive integer n_0 such that if $n>n_0$, then $\sigma(P_nKP_n)$ is contained in the disk of center zero and radius ε. Let $m>n_0$ so that $\|K-P_mK\|<\varepsilon$. Since $P_n\mathfrak{H}$ is finite dimensional there exists a basis $e_1, \dots, e_k$ of P_m on which the representing matrix of the operator $P_mK|P_m\mathfrak{H}$ is in the upper triangular form. Observe that the diagonal elements of this matrix are in absolute value less than ε. Letting $\mathfrak{H}_j$ be the span of the vector e_j, $1\leqq j\leqq k$ and defining $\mathfrak{H}_{k+1}=\mathfrak{H}\ominus P_m\mathfrak{H}$, we deduce that the lower triangular part of K with respect to the decomposition $\mathfrak{H}=\sum_{j=1}^{k+1}\oplus\mathfrak{H}_j$ has norm less than 3ε. Since ε is arbitrary we conclude that $K\in P(\mathfrak{H})$.

As a consequence of the next theorem we shall see that there are operators in $P(\mathfrak{H})$ which are not quasinilpotent.

In the remainder of the paper $\{e_n\ (n=1, 2, \dots)\}$ will be a fixed orthonormal basis of $\mathfrak{H}$. A weighted shift S with positive weights α_n $(n=1, 2, \dots)$ on the basis $\{e_n\}$ is defined by $Se_n=\alpha_n e_{n+1}$ $(n=1, 2, \dots)$.

Theorem 3.5. *Let S be a weighted shift on the basis $\{e_n\}$ with positive weights α_n $(n=1, 2, \dots)$ such that for every $\varepsilon>0$ there exists a positive integer k satisfying $\alpha_{nk}<\varepsilon$, for $n=1, 2, \dots$. Then $S\in P(\mathfrak{H})$.*

Proof. Let $\varepsilon>0$ and let k be a positive integer greater than 1 such that $\alpha_{nk}<\varepsilon$, for $n=1, 2, \dots$. Furthermore, let $\mathfrak{H}_j$ be the span of the vectors e_{j+ik} $(i=0, 1, 2, \dots)$. Then $\mathfrak{H}_j$ is infinite dimensional, $1\leqq j\leqq k$, and the representing matrix of S on the decomposition $\mathfrak{H}=\sum_{j=1}^{k}\oplus\mathfrak{H}_j$ has the form

$$S=\begin{bmatrix} 0 & 0 & \dots\dots\dots & S_k \\ S_1 & 0 & \dots\dots\dots & 0 \\ 0 & S_2 & \dots\dots\dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots\ S_{k-1} & 0 \end{bmatrix},$$

where, for $1\leqq j\leqq k-1$, the bounded, linear transformation $S_j\colon \mathfrak{H}_j\to\mathfrak{H}_{j+1}$ is unitarily equivalent to a diagonal operator in $\mathscr{L}(\mathfrak{H})$ whose diagonal terms are α_{j+ik} $(i=0, 1, 2, \dots)$ and $S_k\colon \mathfrak{H}_k\to\mathfrak{H}_1$ is a bounded linear transformation unitarily equivalent to a weighted shift with weights α_{nk}. Thus $\|S_k\|<\varepsilon$ and hence $S^k\in P(\mathfrak{H})$. Therefore $S\in P(\mathfrak{H})$ and our assertion is established.

Corollary 3.6. *There exists an operator in $P(\mathfrak{H})$ whose spectrum coincides with D and hence is not quasinilpotent.*

Proof. Let S be the Kakutani shift on the basis $\{e_n\}$, i.e. S is the weighted shift whose sequence of weights is described as follows: every other weight is one;

every other weight of the remaining weights is 1/2; every other weight of these weights is 1/4; etc. For the sake of clarity we list the first few terms of the sequence of weights:

$$1, 1/2, 1, 1/4, 1, 1/2, 1, 1/8, 1, 1/2, 1, 1/4, \ldots .$$

From Theorem 3.5 it follows that $S \in P(\mathfrak{H})$. On the other hand, as is well known, KAKUTANI proved that $\sigma(S)=D$ [7, p. 282].

In view of the results of this section and the comment made at the end of Section 2 it is natural to pose the following two questions.

Problem 1. Is every quasinilpotent operator on $\mathfrak{H}$ in $P(\mathfrak{H})$?

Problem 2. Does every operator in $P(\mathfrak{H})$ have a non-trivial invariant subspace?

Addendum: The results proved in the present note were obtained in the Spring of 1973 and were communicated to several mathematicians interested in the subject during the Wabash International Conference on Banach spaces held in June 1973. After this paper was written C. Apostol, C. Foiaş and D. Voiculescu announced that they established the validity of the conjectures stated at the end of section 2. This announcement was recently communicated to the author by C. Apostol via a personal letter.

Added in proof (May 5, 1975). The results referred to in Addendum appeared in C. APOSTOL, C. FOIAŞ and D. VOICULESCU, On the norm closure of nilpotents. II, *Rev. Roum. Math. Pures et Appl.*, **19** (1974), 549—577, and D. VOICULESCU, Norm-limits of algebraic operators, *Rev. Roum. Math. Pures et Appl.*, **19** (1974) 371—378.

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