Characterization of some related semigroups of universal algebras

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§ 1. Introduction

In [3], B. JÓNSSON gave a necessary and sufficient condition for a group of permutations of a set A to be the automorphism group of an (universal) algebra whose base set is A. In [2], G. GRÄTZER characterized those abstract semigroups that are isomorphic to the endomorphism semigroup of some simple algebra. These results are often referred to as the solution of the concrete characterization problem of automorphism groups of algebras and that of the abstract characterization problem of endomorphism semigroups of simple algebras.

In this note we are going to solve the concrete characterization problems of

a) inverse semigroups of partial automorphisms of algebras (Theorem 1), and

b) semigroups of endomorphisms of simple algebras (Theorem 2)

in the above sense*).

Let us consider a set A ($|A|\geqq 2$), which will be fixed in the sequel. By a 1—1 partial transformation of A we mean a 1—1 mapping from a subset of A into A. The semigroup of all 1—1 partial transformations of A, called in [1] the symmetric inverse semigroup of A, will be denoted by I_A. For any $\varphi\in I_A$ let $D(\varphi)$ be the domain of φ, and $\varphi|B$ the restriction of φ to B, where $B\subseteqq A$.

By the equalizer of any $\varphi, \psi\in I_A$ we mean the set $E(\varphi, \psi)$ defined by

$$E(\varphi, \psi) = \langle a \mid a\in D(\varphi)\cap D(\psi) \text{ and } a\varphi = a\psi\rangle.$$

For any $M\subseteqq I_A$ and $B\subseteqq A$, put

$$\Gamma_M(B) = \bigcap\langle E(\varphi, \psi) \mid \varphi, \psi\in M \text{ and } B\subseteqq E(\varphi, \psi)\rangle.$$

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*) *(Added May 23, 1974)* The originally submitted version of the article contained also a solution for the concrete characterization problem of semigroups of 1—1 endomorphisms of algebras. It was omitted as in the meantime the solution of this problem was published by J. JEŽEK in *Coll. Math.*, 29 (1974), 61—69 (Theorem 2).

B. M. SCHEIN kindly informed us that our Theorem 1 was obtained independently also by D. A. BREDIHIN in Saratov.

Then Γ_M is a closure operator and we may speak of a Γ_M-closed subset of A. Note that $D(\varphi)$ is a Γ_M-closed set for any $\varphi \in M$, (indeed, $E(\varphi, \varphi) = D(\varphi)$), and thus $B \subseteq D(\varphi)$ implies $\Gamma_M(B) \subseteq D(\varphi)$. Furthermore, if the identity transformation of A belongs to M then $a \in \Gamma_M(\emptyset)$ if and only if $a \in D(\varphi)$ and $a = a\varphi$ for all $\varphi \in M$.

By a partial automorphism of an algebra (A, F) we mean an isomorphism of a subalgebra of (A, F) into (A, F). The empty transformation $0: \emptyset \to \emptyset$ is considered to be partial automorphism if and only if (A, F) has no nullary operation. $\mathrm{Aut}_p(A, F)$ denotes the set of all partial automorphisms of (A, F).

We shall often write x instead of $(x_1, \ldots, x_n)$ and, similarly, $x\varphi$ instead of $(x_1\varphi, \ldots, x_n\varphi)$ for any mapping φ. Then, $x^{\urcorner}$ stands for the set of all components of any $x \in A^n$. Finally, $\nu(f)$ denotes the arity of the operation f (i.e., f maps $A^{\nu(f)}$ into A).

§ 2. Results

We start with two simple lemmas.

Lemma 1. *For any algebra (A, F), the semigroup $\mathrm{Aut}_p(A, F)$ is an inverse subsemigroup of I_A. Furthermore, (B, F) $(B \subseteq A)$ is a subalgebra of (A, F) if and only if B is a $\Gamma_{\mathrm{Aut}_p(A,F)}$-closed set.*

Proof. The first statement is trivial. Suppose that (B, F) is a subalgebra of (A, F). If $B \neq \emptyset$ then let ε be the identity automorphism of (B, F). Clearly, $\varepsilon \in \mathrm{Aut}_p(A, F)$ and $E(\varepsilon, \varepsilon) = B$. Thus $\Gamma_{\mathrm{Aut}_p(A,F)}(B) = B$. If $B = \emptyset$ then (A, F) has no nullary operation, and thus the empty transformation 0 belongs to $\mathrm{Aut}_p(A, F)$. Then $E(0, 0) = \emptyset$, which implies $\Gamma_{\mathrm{Aut}_p(A,F)}(\emptyset) = \emptyset$. The converse follows from the fact that $(E(\varphi, \psi), F)$ is a subalgebra of (A, F) for any $\varphi, \psi \in \mathrm{Aut}_p(A, F)$.

Lemma 2. *Let M be an inverse subsemigroup of I_A; $\varphi \in M$ and $B \subseteq D(\varphi)$. Then $\Gamma_M(B\varphi) = \Gamma_M(B)\varphi$.*

Proof. From the definition of Γ_M it follows that $u \in \Gamma_M(B)$ $(B \subseteq A)$ if and only if for any $\sigma, \tau \in M$, $B \subseteq D(\sigma) \cap D(\tau)$ and $\sigma | B = \tau | B$ implies $u \in D(\sigma) \cap D(\tau)$ and $u\sigma = u\tau$.

If $\sigma | B\varphi = \tau | B\varphi$ $(\sigma, \tau \in M;\ B\varphi \subseteq D(\sigma) \cap D(\tau))$ then $\varphi\sigma | B = \varphi\tau | B$, and thus $\varphi\sigma | \Gamma_M(B) = \varphi\tau | \Gamma_M(B)$, whence $\sigma | \Gamma_M(B)\varphi = \tau | \Gamma_M(B)\varphi$. Hence, $\Gamma_M(B)\varphi \subseteq \Gamma_M(B\varphi)$. Write $B\varphi$ and φ^{-1} instead of B and φ, respectively. Then we get $\Gamma_M(B\varphi)\varphi^{-1} \subseteq \Gamma_M((B\varphi)\varphi^{-1}) = \Gamma_M(B)$, which implies $\Gamma_M(B\varphi) \subseteq \Gamma_M(B)\varphi$, Q.E.D.

For any $M \subseteq I_A$, we say that $\varphi (\in I_A)$ belongs to M locally if for any finite set $B \subseteq D(\varphi)$ there is a $\psi \in M$ such that $\varphi | B = \psi | B$.

Theorem 1. *Let M be an inverse subsemigroup of I_A, which contains the identity transformation of A. The following two statements are equivalent:*

I. *There is an algebra (A, F) such that $M = \mathrm{Aut}_p(A, F)$.*

II. (α) *Γ_M is an algebraic closure operation,*

(β) *if for $\varphi \in I_A$, $D(\varphi)$ is a Γ_M-closed set and φ belongs to M locally, then $\varphi \in M$,*

(γ) *all 1—1 partial transformations $\varphi\colon \{a\} \to \{b\}$, for which $\{a\}$ and $\{b\}$ are Γ_M-closed sets, belong to M.*

Proof. I⇒II. According to Lemma 1 we get (α) and (γ) immediately. Suppose that $\varphi \in I_A$ satisfies the condition of (β). Since $D(\varphi)$ is a $\Gamma_{\mathrm{Aut}_p(A,F)}$-closed set, by Lemma 1, we have that $(D(\varphi), F)$ is a subalgebra of (A, F). If $D(\varphi) \neq \emptyset$ (i.e., $\varphi \neq 0$), then let $f \in F$ and $x \in A^{\nu(f)} (x^\urcorner \subseteq D(\varphi))$. Then there exists a $\psi \in \mathrm{Aut}_p(A, F)$ which agrees with φ on $x^\urcorner \cup \{f(x)\}$. Thus $f(x\varphi) = f(x\psi) = f(x)\psi = f(x)\varphi$, whence $\varphi \in \mathrm{Aut}_p(A, F)$. If $D(\varphi) = \emptyset$ (i.e., $\varphi = 0$), then by Lemma 1 $(\emptyset, F)$ is a subalgebra of (A, F). Therefore, (A, F) has no nullary operation, and thus $0 \in \mathrm{Aut}_p(A, F)$.

II⇒I. We shall construct the desired algebra (A, F). For any $x = (x_1, \ldots, x_n) \in A^n$ $(n = 1, 2, \ldots)$ and $u \in \Gamma_M(x^\urcorner)$, let $f_{x,u}\colon A^n \to A$ be defined by

$$f_{x,u}(x\varphi) = u\varphi, \quad \text{for all} \quad \varphi \in M,$$

$$f_{x,u}(y) = y_1, \quad \text{if} \quad y = (y_1, \ldots, y_n) \in A^n \setminus xM.$$

From $u \in \Gamma_M(x^\urcorner)$ it follows that the definition of $f_{x,u}$ is correct. Put $F = \{f_{x,u} \mid x \in A^n;\ n = 1, 2, \ldots \text{ and } u \in \Gamma_M(x^\urcorner)\} \cup \Gamma_M(\emptyset)$. (The elements of $\Gamma_M(\emptyset)$ are exactly the nullary operations of (A, F).) We prove that $M = \mathrm{Aut}_p(A, F)$.

Let $\varphi \in M$, $\varphi \neq 0$. First we show that $(D(\varphi), F)$ is subalgebra of (A, F). It is clear that $\Gamma_M(\emptyset) \subseteq D(\varphi)$. Let $f_{x,u} \in F$ and $y \in A^{\nu(f_{x,u})}$, $y^\urcorner \subseteq D(\varphi)$. If $y = x\psi$ for some $\psi \in M$, then $f_{x,u}(y) = f_{x,u}(x\psi) = u\psi$. By Lemma 2, $u \in \Gamma_M(x^\urcorner)$ implies $u\psi \in \Gamma_M(x^\urcorner\psi) = \Gamma_M(y^\urcorner)$. But $y^\urcorner \subseteq D(\varphi)$, and thus $u\psi \in \Gamma_M(y^\urcorner) \subseteq \Gamma_M(D(\varphi)) = D(\varphi)$. If $y \in A^{\nu(f_{x,u})} \setminus xM$, then $f_{x,u}(y) = y_1 \in D(\varphi)$. Hence, $D(\varphi)$ is closed under $f_{x,u}$. To prove that φ is an isomorphism let $f_{x,u} \in F$ and $y \in A^{\nu(f_{x,u})}$, $y^\urcorner \subseteq D(\varphi)$. If y can be written in the form $x\psi$ $(\psi \in M)$, then $f_{x,u}(y\varphi) = f_{x,u}((x\psi)\varphi) = f_{x,u}(x(\psi\varphi)) = u(\psi\varphi) = (u\psi)\varphi = f_{x,u}(x\psi)\varphi = f_{x,u}(y)\varphi$. If $y \in A^{\nu(f_{x,u})} \setminus xM$, then $y\varphi \in A^{\nu(f_{x,u})} \setminus xM$, and thus $f_{x,u}(y\varphi) = y_1\varphi = f_{x,u}(y)\varphi$. It is evident that all elements of $\Gamma_M(\emptyset)$ remain fixed under φ. If the empty transformation belongs to M, then $\Gamma_M(\emptyset) = \emptyset$, i.e., (A, F) has no nullary operation. Thus $0 \in \mathrm{Aut}_p(A, F)$.

Suppose that $\varphi \in I_A$, $\varphi \neq 0$, but $\varphi \notin M$. If $D(\varphi)$ is not a Γ_M-closed set, then $(D(\varphi), F)$ is not a subalgebra of (A, F). To show this statement let $u \in \Gamma_M(D(\varphi))$ and $u \notin D(\varphi)$. Since Γ_M is an algebraic closure operator thus there is a finite set $B \subseteq D(\varphi)$ such that $u \in \Gamma_M(B)$. Arrange the elements of B into a one-to-one sequence x. Then $f_{x,u}(x) = u$ showing that $D(\varphi)$ is not closed under $f_{x,u}$.

If $D(\varphi)$ is a Γ_M-closed set then, by (β), there exists a finite set $B \subseteqq D(\varphi)$ such that no element of M agrees with φ on B. If $|D(\varphi)| \geqq 2$, then we can assume that $|B| \geqq 2$. Arrange the elements of B into a one-to-one sequence x. Then $f_{x,x_2}(x\varphi) = x_1\varphi \neq x_2\varphi = f_{x,x_2}(x)\varphi$. If $|D(\varphi)|=1$, i.e., $\varphi\colon \{a\} \to \{b\}$ for some $a, b \in A$, then by (γ), $\{b\}$ is not a Γ_M-closed set. Thus there is a $u \in \Gamma_M(\{b\})$ such that $u \neq b$. Furthermore, a cannot be written in the form $a = b\psi$ ($\psi \in M$) as, by Lemma 2, from $a = b\psi$ we get $\Gamma_M(\{b\}) = \Gamma_M(\{a\}\psi^{-1}) = \Gamma_M(\{a\})\psi^{-1} = \{a\}\psi^{-1} = \{b\}$, which is a contradiction. Thus $f_{b,u}(a\varphi) = f_{b,u}(b) = u \neq b = a\varphi = f_{b,u}(a)\varphi$.

If $0 \notin M$ then, by (β), $\Gamma_M(\emptyset) \neq \emptyset$, i.e., (A, F) has nullary operation. Thus $0 \notin \mathrm{Aut}_p(A, F)$. Q.E.D.

For any $M \subseteqq I_A$, the inverse subsemigroup of I_A generated by M is denoted by $\tilde{M}$. Further, for any transformation semigroup S of A, the images of the constant transformations of S will be referred to as the constants of S.

Theorem 2. *For any transformation monoid S of A, the following two statements are equivalent:*

I. *There exists a simple algebra (A, F) such that $S = \mathrm{End}\,(A, F)$.*

II. (α) *$S = M \cup C$, where M contains only 1—1 and C contains only constant transformations,*

(β) *if a 1—1 transformation φ of A belongs to $\tilde{M}$ locally, then $\varphi \in M$,*

(γ) *the set of all constants of S is closed under any $\varphi \in \tilde{M}$,*

(δ) *all $a \in A$ such that $\{a\}$ is a Γ_M-closed set are constants of S.*

Proof. I$\Rightarrow$II. (α) is trivial, (β) follows from Theorem 1, (γ) is valid because the product of homomorphisms is also a homomorphism, and (δ) follows from the fact that $(\{a\}, F)$ is a subalgebra of (A, F) whenever $\{a\}$ is a $\Gamma_{\tilde{M}}$-closed subset in A.

II$\Rightarrow$I. We construct the desired algebra (A, F). For any $x = (x_1, \dots, x_n) \in A^n$ ($n = 2, 3, \dots$) let $f_x\colon A^n \to A$ be defined by

$$f_x(x\varphi) = x_2\varphi, \quad \text{for all} \quad \varphi \in \tilde{M},$$

$$f_x(y) = y_1, \quad \text{if} \quad y = (y_1, \dots, y_n) \in A^n \setminus x\tilde{M}.$$

Furthermore, for any $a \in A$ such that a is not a constant of S and $u \in \Gamma_{\tilde{M}}(\{a\})$, let $h_{a,u}\colon A \to A$ be defined by

$$h_{a,u}(a\varphi) = u\varphi, \quad \text{for all } \varphi \in \tilde{M}$$

$$h_{a,u}(x) = x, \quad \text{if} \quad x \in A \setminus a\tilde{M}.$$

Let F be the set of all operations of form f_x as well as $h_{a,u}$. We shall prove that $S = \mathrm{End}\,(A, F)$ and (A, F) is a simple algebra.

Let $\varphi\in S$. If $\varphi\in M$, then to prove that φ commutes with all operations of F we may proceed similarly as we put it in the proof of Theorem 1. If $\varphi\in C$, i.e., $\varphi\colon A\to\{d\}$ $(d\in A)$, then φ commutes with any $f_x\in F$ because f_x is an idempotent operation. Let $h_{a,u}\in F$ and $x\in A$. Then $d\in A\setminus a\tilde{M}$ because from $a\psi=d(\psi\in\tilde{M})$ we get $a=d\psi^{-1}$ and, by (γ), this implies that a is a constant of S, which is a contradiction. Thus $h_{a,u}(x\varphi)=h_{a,u}(d)=d=h_{a,u}(x)\varphi$.

Let φ be a transformation of A and $\varphi\notin S$. If φ is one-to-one then, by (β), for some $n(\geqq 2)$ there exists an $x\in A^n$ such that $x\varphi\in A^n\setminus x\tilde{M}$. Thus $f_x(x\varphi)=x_1\varphi\neq \neq x_2\varphi=f_x(x)\varphi$. If φ is a constant transformation, i.e., $\varphi\colon A\to\{d\}$ $(d\in A)$, then d is not a constant of S, and thus, by (δ), we have $\Gamma_{\tilde{M}}(\{d\})\neq\{d\}$. Therefore, for a suitable $u\in\Gamma_{\tilde{M}}(\{d\})$ we get $u\neq d$. Then $h_{d,u}(d\varphi)=h_{d,u}(d)=u\neq d=h_{d,u}(d)\varphi$. If φ is neither 1—1 nor a constant transformation, then there are x_1, x_2, x_3 and x_4 in A with $x_3\neq x_4$ such that $x_1\varphi\neq x_2\varphi$ and $x_3\varphi=x_4\varphi$. Put $x=(x_1, x_2, x_3, x_4)$. It is clear that $x\varphi\in A^4\setminus x\tilde{M}$, and thus $f_x(x\varphi)=x_1\varphi\neq x_2\varphi=f_x(x)\varphi$.

Now we have to prove only that (A, F) is simple algebra. Let Θ be a congruence relation of (A, F), and suppose that $a\equiv b(\Theta)$ for some $a, b\in A$, $a\neq b$. We claim that $c\equiv d(\Theta)$ for all $c, d\in A$. Put $x=(c, d, a, a)$ and $y=(c, d, a, b)$. Since a and b are distinct thus, $y\in A^4\setminus x\tilde{M}$, whence $d=f_x(x)\equiv f_x(y)=c(\Theta)$ follows. Q.E.D.

Remark. It can be shown without any difficulty that none of the conditions (α), (β) and (γ) in Theorem 1 is implied by the two others; a similar statement is valid for the conditions (β), (γ) and (δ) in Theorem 3.

References

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