Derivations of lattices

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1. Introduction. A mapping $a \to a'$ of a ring R into itself is called a *derivation* of R if the equations

$$(a+b)' = a'+b', \quad (ab)' = a'b+ab'$$

hold for any pair a, b of R. As a generalization of this definition it offers itself the following one: A mapping $a \to a'$ of an algebra A with two (arbitrary) binary operations $+, \cdot$ into itself is called a derivation of A if (1) and (2) are true for any elements a, b of A.

In this note we investigate the derivations of lattices with the aid of our earlier results in [2] concerning translations of lattices. For the concepts not defined here see [1] or [3].

2. Preliminaries. According to what have been said in the introduction we introduce the following

Definition 1. A single-valued mapping φ of a lattice L into itself is called a *derivation* of L if

(1) $\varphi(x \smile y) = \varphi(x) \smile \varphi(y)$ and $\varphi(x \frown y) = (\varphi(x) \frown y) \smile (x \frown \varphi(y))$

for every pair of elements x, y of L.

Examples:

1. In every lattice L, the identity mapping ι defined by $\iota(x)=x$ for each $x \in L$ is a derivation of L.

2. Let L be a lattice with least element o. Then the mapping ω defined by $\omega(x)=o$ for each $x \in L$ is a derivation of L.

3. To every neutral element n of a lattice L there corresponds a derivation generated by n, namely the mapping φ_n defined by $\varphi_n(x)=n \frown x$ for each $x \in L$.

In [2] we defined the translations of a lattice and established their basic properties. In studying the derivations we shall need the dual concept. Therefore we distinguish now join-translations and meet-translations as follows:

Definition 2. A single-valued mapping λ of a lattice L into itself is called a *join-translation* if

$$\lambda(x \smile y) = \lambda(x) \smile y \tag{2}$$

and a *meet-translation* if

$$\lambda(x \frown y) = \lambda(x) \frown y \tag{3}$$

for each pair of elements x, y of L.

It was shown in [2] that the only mapping of L into itself which is a join-translation as well as a meet-translation is the identity mapping of L.

For sake of completeness of this note we formulate all those results of [2] that will be applied here.

Proposition 1. *Every meet-translation of a lattice L is an idempotent meet-endomorphism* (that is, a meet-endomorphism λ for which $\lambda(\lambda(x)) = \lambda(x)$ identically).

Proposition 2. *The fixed elements of a meet-translation λ of a lattice L form an ideal I_λ of L and, for any two meet-translations λ_1, λ_2 of L, $\lambda_1 \neq \lambda_2$ implies $I_{\lambda_1} \neq I_{\lambda_2}$.*

Proposition 3. *Any two meet-translations of a lattice are permutable.*

Proposition 4. *A lattice L is distributive if and only if every meet-translation of L is (not only a meet-endomorphism, but) an endomorphism of L.*

3. Relations between the class of derivations and other classes of lattice mappings. Every derivation of a lattice L is a join-endomorphism of L, by definition; we show that it is a meet-endomorphism, too, by proving the

Theorem 1. *Every derivation of a lattice L is a meet-translation of L.*

Corollary 1. *Every derivation of a lattice L is an idempotent endomorphism of L.*

Corollary 2. *The fixed elements of a derivation of a lattice L form an ideal of L and the ideal of fixed elements determines uniquely the derivation in question.*

We reach to Theorem 1 by proving the following lemmas concerning any derivation φ of a lattice L:

Lemma 1. *$\varphi(x) \leqq x$ for any element x of L.*

Lemma 2. *$x \leqq y$ implies $\varphi(x) \leqq \varphi(y)$ $(x, y \in L)$.*

Lemma 3. *$x \leqq y$ implies $\varphi(x) = x \frown \varphi(y)$ $(x, y \in L)$.*

Proof. We have by (1)

$$\varphi(x) = \varphi(x \frown x) = (\varphi(x) \frown x) \smile (x \frown \varphi(x)),$$

i.e. $\varphi(x) = \varphi(x) \frown x$ for any element x of L which is equivalent to the assertion of Lemma 1.

If $x \leqq y$, then (1) implies

$$\varphi(y) = \varphi(x \smile y) = \varphi(x) \smile \varphi(y),$$

i.e. $\varphi(x) \leqq \varphi(y)$, as asserted in Lemma 2.

Let $x \leqq y$ again. Then $\varphi(x) \leqq x \leqq y$ by Lemma 1. Consequently

$$\varphi(x) = \varphi(x \frown y) = (\varphi(x) \frown y) \smile (x \frown \varphi(y)) = \varphi(x) \smile (x \frown \varphi(y)),$$

i.e. $x \frown \varphi(y) \leqq \varphi(x)$. On the other hand, $\varphi(x) \leqq x$ by Lemma 1 and $\varphi(x) \leqq \varphi(y)$ by Lemma 2. Therefore

$$x \frown \varphi(y) \geqq \varphi(x),$$

too, completing the proof of Lemma 3.

Applying Lemma 3 to the case $x = u \frown v$, $y = u$ we get

$$\varphi(u \frown v) = u \frown v \frown \varphi(u) = \varphi(u) \frown v$$

for any elements u, v of L (since $u \frown \varphi(u) = \varphi(u)$ by Lemma 1). Thus every derivation of L identically satisfies (3) and therefore it is a meet-translation of L, indeed.

Corollary 1 follows from Definition 1 and Proposition 1. Corollary 2 follows from Proposition 2. Thus Theorem 1 and the corollaries following it have been proved.

As a simple consequence of Lemma 3 we have also the

Corollary 3. *Every derivation φ of a lattice L is of the form*

$$\varphi(x) = c \frown x \tag{4}$$

with a suitably chosen $c \in L$ if and only if L has a greatest element.

Proof. If i is the greatest element of L, then $\varphi(x) = x \frown \varphi(i)$ for each $x \in L$, by Lemma 3. If, however, L has no greatest element, then the identity mapping of L cannot be represented in the form (4), because $c \frown x \neq x$ for $x > c$.

It is easy to see that *the class of all derivations of a lattice L at least of two elements is a proper subclass of all endomorphisms of L.* In fact, given an element c different from the least element (eventually existing) of L, the mapping $\gamma(x)$ define by

$$\gamma(x) = c \quad \text{for each} \quad x \in L$$

is an endomorphism of L which is no derivation because there exists at least one element d in L such that $c>d$ and thus $\gamma(d)>d$. Hence Lemma 1 does not hold for this mapping γ.

Now we are going to give a fuller characterization of the derivations among the meet-translations and the endomorphisms.

Theorem 2. *Let D, T, J, E denote the set of all derivations, meet-translations, join-endomorphisms and endomorphisms, respectively, of a lattice L. Then*

$$D = T \cap J = T \cap E.$$

In other words, a meet-translation of a lattice is a derivation if and only if it is a join-endomorphism (or equivalently, an endomorphism) of that lattice.

Corollary 4. *Let I be an ideal of the lattice L and φ an endomorphism of L onto I such that $\varphi(x)=x$ for each $x \in I$. Then φ is a derivation of L.*

Corollary 5. *A lattice is distributive if and only if $D=T$.*

Proof. $D \subseteqq T$ by Theorem 1 and $D \subseteqq J$ by Definition 1. Therefore $D \subseteqq T \cap J$.

On the other hand, the second *equation* (1) *is identically satisfied by any meet-translation of a lattice.* For, if φ is a meet-translation of the lattice L, then

$$\varphi(x \frown y) = \varphi(x) \frown y \quad \text{and} \quad \varphi(x \frown y) = \varphi(y \frown x) = \varphi(y) \frown x = x \frown \varphi(y)$$

for any elements x, y of L whence the second equation (1) trivially follows. This means that any mapping $\varphi \in T \cap J$ satisfies (1). Consequently, $T \cap J \subseteqq D$. Thus the equation $D = T \cap J$ has been verified.

Let M denote the set of all meet-endomorphisms of L. Then, by Proposition 1, $T = T \cap M$. Hence $T \cap J = T \cap M \cap E = T \cap E$, completing the proof of Theorem 2.

Now, let φ be an endomorphism of L that satisfies the conditions in Corollary 4. Since I is, a fortiori, an ideal of the meet-semilattice $L^\frown$ of L, the mapping φ is a translation of $L^\frown$ by Theorem 2 of [3]. Hence, φ is (not only an endomorphism but) a meet-translation of L.

Corollary 5 is an immediate consequence of Theorem 2 and Proposition 4.

Remark. One can derive also immediately from Lemma 3 that every derivation is idempotent (by taking $x=\varphi(t)$ and $y=t$) and that the fixed elements form an ideal (by taking $x \leqq y = \varphi(y)$).

4. Basic properties of the multiplication of derivations. By the product $\varrho\sigma$ of two mappings ϱ and σ of a set S into itself we mean, as usual, the mapping π defined by $\pi(x) = \varrho(\sigma(x))$ $(x \in S)$.

Theorem 3. *The set of all derivations of a given lattice forms a commutative sermigoup with respect to the multiplication of mappings.*

Proof. It is well-known that the multiplication of mappings is associative. Furthermore, any two derivations of a lattice are permutable by Theorem 1 and Proposition 3. Thus we have only to show that the product of any two derivations of a lattice is again a derivation of that lattice.

Let φ and ψ be arbitrary derivations, x and y arbitrary elements of a lattice L. Then, by the first equation (1) we have

$$\varphi\psi(x\smile y) = \varphi(\psi(x)\smile\psi(y)) = \varphi\psi(x)\smile\varphi\psi(y),$$

that is, $\varphi\psi$ is a join-endomorphism of L. Moreover, by both equations (1) we get

$$\varphi\psi(x\frown y) = \varphi((\psi(x)\frown y)\smile(x\frown\psi(y))) = \varphi(\psi(x)\frown y)\smile\varphi(x\frown\psi(y)),$$

whence, by the second equation (1),

$$\varphi\psi(x\frown y) = (\varphi\psi(x)\frown y)\smile(\psi(x)\frown\varphi(y))\smile(\varphi(x)\frown\psi(y))\smile(x\frown\varphi\psi(y)). \tag{5}$$

In order to prove the theorem we have to show that the right-hand side of (5) reduces to

$$(\varphi\psi(x)\frown y)\smile(x\frown\varphi\psi(y)).$$

We shall achieve this purpose by verifying the inequalities

$$\psi(x)\frown\varphi(y) \leqq x\frown\varphi\psi(y), \tag{6}$$

$$\varphi(x)\frown\psi(y) \leqq \varphi\psi(x)\frown y. \tag{7}$$

By Corollary 1, φ and ψ are endomorphisms of L. Therefore

$$\varphi\psi(x\frown y) = \varphi\psi(x)\frown\varphi\psi(y).$$

Combining this equation with (5) we see that

$$\psi(x)\frown\varphi(y) \leqq \varphi\psi(x)\frown\varphi\psi(y). \tag{8}$$

Since $\varphi\psi(x)\leqq\psi(x)\leqq x$ by Lemma 1, (8) implies (6). Inequality (7) can be derived similarly.

5. On the fixed ideal and the kernel of a derivation. By Theorem 1 and Proposition 2, the fixed elements of a derivation φ (that is, the elements x such that $\varphi(x)=x$) of a lattice L form an ideal of L. This ideal will be called the *fixed ideal* of φ and denoted by Fix φ.

Let L be a lattice with least element o. Then, by the *kernel* of φ we mean the set of all elements x of L such that $\varphi(x)=o$. The kernel of φ will be denoted by Ker φ.

We make some comments on the relations between the fixed ideal and the kernel of a derivation φ of a lattice L with least element o.

Remark 1. $\mathrm{Fix}\,\varphi\cap\mathrm{Ker}\,\varphi=\{o\}$.

Remark 2. Fix $\varphi=\{o\}$ implies Ker $\varphi=L$.

Remark 3. If L has at least two elements, then Ker $\varphi=\{o\}$ implies Fix $\varphi\supset\{o\}$.

Remark 4. There exist lattices L with least element o such that Ker $\varphi\supset\{o\}$ and Fix $\varphi\supset\{o\}$.

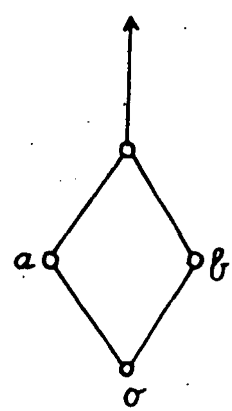
Proof.

1. If $x\in$ Fix $\varphi\cap$ Ker φ, then $x=\varphi(x)=o$.

2. By Corollary 1, $\varphi(x)\in$ Fix φ for each $x\in L$. Thus, Fix $\varphi=\{o\}$ implies that $\varphi(x)=o$ for each $x\in L$.

3. If L has at least two elements, then there exists an element $c\in L$ such that $c\neq o$. Suppose Ker $\varphi=\{o\}$. Then $\varphi(c)\neq o$ and $\varphi(c)\in$ Fix φ, by Corollary 1.

4. Consider the lattice of the diagram below where the arrow directed upward denotes an arbitrary chain (with or without a greatest element). Then the mapping



defined by $\varphi(x)=a\frown x$ is a derivation of the lattice for which Fix $\varphi=(a]$ and Ker $\varphi=(b]$.

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