Factorization of operators in $\mathscr{C}_\varrho$ classes

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1. Let T be a linear bounded operator on a Hilbert space $\mathfrak{H}$ and ϱ a positive number. If U is a unitary operator on a Hilbert space $\mathfrak{K}\supset\mathfrak{H}$ and

$$T^n h = \varrho P U^n h \quad \text{for} \quad h\in\mathfrak{H}, \qquad n = 1, 2, \dots,$$

where P (as always in the following) is the orthogonal projection onto $\mathfrak{H}$, then we say that U is a unitary ϱ-dilation of T. $\mathscr{C}_\varrho$ denotes the class of those operators which have a unitary ϱ-dilation.

The study of $\mathscr{C}_\varrho$ classes and unitary ϱ-dilations was initiated by B. SZ.-NAGY and C. FOIAŞ [3] and continued by a number of authors. Recently T. ANDO [1] proved that $T\in\mathscr{C}_2$ if and only if there exists a contraction C on $\mathfrak{H}$ such that

$$T = 2(I-C^*C)^{1/2}C.$$

Moreover, using this factorization, he constructed a unitary 2-dilation of T on $\bigoplus_{n=-\infty}^{\infty} \mathfrak{H}_n$ $(\mathfrak{H}_n=\mathfrak{H})$ by a matrix of operator entries.

Our purpose is to generalize Ando's results for $\varrho>0$, $\varrho\neq 1$. ($\mathscr{C}_1$ is the class of contractions, cf. [*], Ch.I). Although we shall not explicitly construct a matrix representation of the unitary ϱ-dilation for $T\in\mathscr{C}_\varrho$, we do construct in Proposition 2 an operator-matrix representation of a contractive ϱ-dilation of T. Since an operator-matrix representation of the unitary 1-dilation of a contraction is well known ([2]; [*], Ch.I) it is only a matter of computation to combine the two representations to obtain a matrix representation of a unitary ϱ-dilation of T.

2. Suppose U is a unitary ϱ-dilation of T on $\mathfrak{K}$. Consider the subspaces

$$\mathfrak{L}_0 = \bigvee_{n=0}^{\infty} U^{-n}\mathfrak{H}, \quad \mathfrak{L}_1 = \mathfrak{L}_0 \vee U\mathfrak{L}_0$$

and denote by Q the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{L}_0$.

For $n=0, 1, \ldots$ and $h, g \in \mathfrak{H}$ we have

$$(QU(QU-T)h, U^{-n}g) = (UQ(U-T)h, U^{-n}g) = ((U-T)h, U^{-n-1}g) =$$
$$= (U^{n+2}h, g) - (U^{n+1}Th, g) = \frac{1}{\varrho}(T^{n+2}h, g) - \frac{1}{\varrho}(T^{n+1}Th, g) = 0.$$

Hence $QU(QU-T)h$ is orthogonal to $\mathfrak{L}_0$; as on the other hand it is contained in $\mathfrak{L}_0$, we have

(1) $$QU(QU-T)h=0 \quad \text{for} \quad h \in \mathfrak{H}.$$

Also notice that as $\mathfrak{H} \subset \mathfrak{L}_0$ we have

(2) $$PQUh = PUh = \frac{1}{\varrho} Th \quad \text{for} \quad h \in \mathfrak{H}.$$

Now we can prove:

Lemma. *For an arbitrary $h \in \mathfrak{H}$,*

$$\inf_{g \in \mathfrak{H}} \{\|g-(QU-T)h\|^2 - \|QUg\|^2\} = 0.$$

Proof. Because $\mathfrak{L}_1 = \mathfrak{L}_0 \vee U\mathfrak{H}$, we have

(3) $$\mathfrak{L}_1 = \mathfrak{L}_0 \oplus \overline{(I-Q)U\mathfrak{H}}.$$

For $h \in \mathfrak{H}$ we have $U(QU-T)h \in \mathfrak{L}_1$ and by (1) $U(QU-T)h \perp \mathfrak{L}_0$, consequently

(4) $$U(QU-T)h \in \mathfrak{L}_1 \ominus \mathfrak{L}_0.$$

So, using (4) and (3), we conclude that there exists a sequence $g_n \in \mathfrak{H}$ such that

(5) $$U(QU-T)h = \lim_{n\to\infty} (I-Q)Ug_n.$$

Now, again by (4) and by (1), for $g, h \in \mathfrak{H}$ we have

$$0 \leqq \|(I-Q)Ug - U(QU-T)h\|^2 = \|(I-Q)U(g-(QU-T)h)\|^2 =$$
$$= \|U(g-(QU-T)h)\|^2 - \|QU(g-(QU-T)h)\|^2 = \|g-(QU-T)h\|^2 - \|QUg\|^2.$$

Setting g_n for g, on account of (5) this proves the assertion of the Lemma.

Now we are going to prove our main assertions.

3. Proposition 1. *If $T \in \mathscr{C}_\varrho$ ($\varrho > 0$, $\varrho \neq 1$) then there exists a contraction C on $\mathfrak{H}$ such that*

$$T = \varrho\big(I + \varrho(\varrho-2)C^*C\big)^{-1/2}(I - C^*C)^{1/2}C.$$

Proof. Suppose $T \in \mathscr{C}_\varrho$. Then using the above notations set

$$Z = (PU^*QUP|\mathfrak{H})^{1/2}.$$

Clearly Z is a non-negative contraction and

(6) $$\|Zh\| = \|QUh\| \quad \text{for} \quad h \in \mathfrak{H}.$$

Lemma and (2) imply that

$$0 = \inf_{g\in\mathfrak{H}} \{\|g-(QU-T)h\|^2 - \|QUg\|^2\} =$$

$$= \inf_{g\in\mathfrak{H}} \left\{\|g\|^2 - 2\left(\frac{1}{\varrho}-1\right)\operatorname{Re}(g, Th) + \|QUh\|^2 + \left(1-\frac{2}{\varrho}\right)\|Th\|^2 - \|QUg\|^2\right\}.$$

Using (6) and denoting

(7) $$Y=(I-Z^2)^{1/2}$$

the above equality can be converted to

(8) $$\inf_{g\in\mathfrak{H}} \left\{\|Yg\|^2 - 2\left(\frac{1}{\varrho}-1\right)\operatorname{Re}(g, Th) + \|Zh\|^2 + \left(1-\frac{2}{\varrho}\right)\|Th\|^2\right\} = 0$$

for every $h\in\mathfrak{H}$.

We are going to prove that

(9) $$\|T^*g\| \leqq M\|Yg\|$$

for every $g\in\mathfrak{H}$ with a suitable positive M independent of g. Suppose in the contrary that for every positive M there exists $g_M\in\mathfrak{H}$ such that

$$\|T^*g_M\| > M\|Yg_M\|.$$

Now apply (8) with $T^*g_M \operatorname{sgn}\left(\frac{1}{\varrho}-1\right)$ in place of h. Setting Mg_M for g we get

$$0 \leqq M^2\|Yg_M\|^2 - 2M\left|\frac{1}{\varrho}-1\right|\|T^*g_M\|^2 + \|ZT^*g_M\|^2 + \left(1-\frac{2}{\varrho}\right)\|TT^*g_M\|^2 <$$

$$< \left(1-2M\left|\frac{1}{\varrho}-1\right| + \|Z\|^2 + \left|1-\frac{2}{\varrho}\right|\|T\|^2\right)\|T^*g_M\|^2 < 0$$

if M is large enough and this is a contradiction.

(9) guarantees the existence of a bounded linear operator X defined everywhere on $\mathfrak{H}$ such that

(10) $$T^*g = XYg \quad \text{if} \quad g\in\mathfrak{H}, \qquad Xf=0 \quad \text{if} \quad f\in\mathfrak{H}\ominus Y\mathfrak{H}.$$

Now (8) implies that

$$0 = \inf_{g\in\mathfrak{H}} \left\{\|Yg\|^2 - 2\left(\frac{1}{\varrho}-1\right)\operatorname{Re}(XYg, h) + \|Zh\|^2 + \left(1-\frac{2}{\varrho}\right)\|YX^*h\|^2\right\} =$$

$$= \inf_{g\in\mathfrak{H}} \left\{\|Yg\|^2 - 2\left(\frac{1}{\varrho}-1\right)\operatorname{Re}(Yg, X^*h) + \left(\frac{1}{\varrho}-1\right)^2\|X^*h\|^2 + \|Zh\|^2 - \right.$$

$$\left. - \frac{1}{\varrho^2}\|X^*h\|^2 + \left(\frac{2}{\varrho}-1\right)\|ZX^*h\|^2\right\}.$$

This means that for every $h \in \mathfrak{H}$

$$\|Zh\|^2 - \frac{1}{\varrho^2}\left(\|X^*h\|^2 + \varrho(\varrho-2)\|ZX^*h\|^2\right) + \inf_{g \in \mathfrak{H}} \left\{ \left\| Yg - \left(\frac{1}{\varrho} - 1\right) X^*h \right\|^2 \right\} = 0. \tag{11}$$

For arbitrary $f \in \mathfrak{H} \ominus Y\mathfrak{H}$ (10) implies that $f \perp X^*\mathfrak{H}$. This shows that $\mathfrak{H} \ominus Y\mathfrak{H} \subset \mathfrak{H} \ominus X^*\mathfrak{H}$ and consequently $\overline{X^*\mathfrak{H}} \subset \overline{Y\mathfrak{H}}$. So we conclude for arbitrary $h \in \mathfrak{H}$ that

$$\inf_{g \in \mathfrak{H}} \left\| Yg - \left(\frac{1}{\varrho} - 1\right) X^*h \right\|^2 = 0.$$

This fact together with (11) imply that

$$\|Zh\| = \frac{1}{\varrho}\left(\|X^*h\|^2 + \varrho(\varrho-2)\|ZX^*h\|^2\right)^{1/2}. \tag{12}$$

Since Z is a non-negative contraction we have

$$((I+\varrho(\varrho-2)Z^2)h, h) \geqq \begin{cases} \|h\|^2 & \text{if } \varrho \geqq 2 \\ (\varrho-1)^2\|h\|^2 & \text{if } 0 < \varrho < 2, \end{cases}$$

consequently for $\varrho > 0$, $\varrho \neq 1$ there exists the positive, boundedly invertible operator $(I+\varrho(\varrho-2)Z^2)^{1/2}$ and, by (12),

$$\|Zh\| = \frac{1}{\varrho}\left\|(I+\varrho(\varrho-2)Z^2)^{1/2}X^*h\right\|.$$

So there exists an operator W on $\mathfrak{H}$ such that

$$\|WZh\| = \|Zh\| \quad \text{for} \quad h \in \mathfrak{H}, \qquad Wf = 0 \quad \text{for} \quad f \in \mathfrak{H} \ominus Z\mathfrak{H}, \tag{13}$$

and

$$WZh = \frac{1}{\varrho}(I+\varrho(\varrho-2)Z^2)^{1/2}X^*h.$$

Now (10) and the invertibility of $(I+\varrho(\varrho-2)Z^2)^{1/2}$ imply that:

$$T = \varrho Y(I+\varrho(\varrho-2)Z^2)^{-1/2}WZ.$$

Let $C = WZ$, then by (7) and (13) we can conclude

$$T = \varrho(I+\varrho(\varrho-2)C^*C)^{-1/2}(I-C^*C)^{1/2}C.$$

Proposition 2. *Suppose C is a contraction on the Hilbert space $\mathfrak{H}$ and $\varrho > 0$, $\varrho \neq 1$. Set*

$$A = (I+\varrho(\varrho-2)C^*C)^{-1/2}, \quad B = (I-C^*C)^{1/2}, \quad B' = (I-CC^*)^{1/2}, \quad T = \varrho ABC,$$

and let V be the linear operator defined on $\mathfrak{H} \oplus \mathfrak{H}$ by the matrix of operator entries

$$\begin{bmatrix} ABC & ABB' \\ (\varrho-1)CAC & (\varrho-1)CAB' \end{bmatrix}.$$

Then: (i) *V is a contraction,*

(ii) $V^2h=VTh$ *for* $h\in\mathfrak{H}$,

(iii) $T^nh=\varrho PV^nh$ *for* $h\in\mathfrak{H}$, *and* $n=1,2,\ldots$,

(iv) $T\in\mathscr{C}_\varrho$.

Proof. Observe that C^*C, A, B commute, $B'C=CB$, $C^*B'=BC^*$, and

$$A^2(B^2+(\varrho-1)^2C^*C)=I.$$

Using these facts an easy computation shows that

$$V^*V=\begin{bmatrix}C^*C & C^*B'\\ B'C & B'^2\end{bmatrix},\quad I-V^*V=\begin{bmatrix}B^2 & -BC^*\\ -CB & CC^*\end{bmatrix}=\begin{bmatrix}B & 0\\ 0 & -C\end{bmatrix}\begin{bmatrix}I & I\\ I & I\end{bmatrix}\begin{bmatrix}B & 0\\ 0 & -C^*\end{bmatrix}\geqq 0,$$

and consequently, $\|V\|\leqq 1$.

The following computation proves (ii): For $h\in\mathfrak{H}$

$$V(V-T)\begin{bmatrix}h\\ 0\end{bmatrix}=V\begin{bmatrix}(1-\varrho)ABCh\\ (\varrho-1)CACh\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}.$$

From (ii) we deduce

$$V^nh=VT^{n-1}h \quad\text{for}\quad h\in\mathfrak{H},\quad n=1,2,\ldots. \tag{14}$$

For $n=1$, (iii) is an immediate consequence of the definition of V and T, and the general case then follows using (14).

Now by virtue of (iii), every unitary 1-dilation U_1 of the contraction V is a ϱ-dilation of T. So (iv) is proved.

By virtue of Propositions 1 and 2 we have:

Theorem. *Suppose $\varrho>0$ and $\varrho\neq 1$. An operator T on $\mathfrak{H}$ belongs to $\mathscr{C}_\varrho$ if and only if there exists a contraction C on $\mathfrak{H}$ such that*

$$T=\varrho(I+\varrho(\varrho-2)C^*C)^{-1/2}(I-C^*C)^{1/2}C.$$

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References

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