Power-bounded operators with invertible characteristic function

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Abstract: Invertibility of the generalized characteristic function of a power-bounded operator is considered. It is proved that a power-bounded operator whose (not necessarily bounded) characteristic function is invertible and whose spectrum has zero Lebesgue measure is a unitary operator. AMS subject classification number: Primary 47A99.

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In this note we wish to consider invertibility of the generalized characteristic function of a power-bounded operator. It is neither surprising nor difficult to see that when the characteristic function is bounded on the open unit disk, the condition for its invertibility is the same as in the case of a contraction. However without any condition on the boundedness of the characteristic function its invertibility always makes the operator similar to a unitary operator. We prove that if in addition the spectrum of such an operator has Lebesgue measure zero then it is in fact a unitary operator.

We consider an operator T on a (separable) Hilbert space $\mathfrak{H}$ and following the notation in [1] we denote the characteristic function of T by $\Theta_T(\lambda)$. Throughout the course of this paper we assume that $\Theta_T(\lambda)$ is invertible whenever $|\lambda|<1$ and write the analytic function $\Omega_T(\lambda)=\Theta_T(\lambda)^{-1}=\sum_{n=0}^{\infty}\lambda^n\omega_n$ where each ω_n is an operator from $\mathfrak{D}_{T^*}$ to $\mathfrak{D}_T$. If $\sup_{|\lambda|<1}\|\Omega_T(\lambda)\|<\infty$ then it follows that, for almost all t, $\Omega_T(re^{it})$ converges strongly to $\Omega_T(e^{it})$ as $r\to 1$, *cf.* [3], Chapter V. Moreover, for u in $H^2(\mathfrak{D}_{T^*})$, $(\Omega_T u)(\lambda)=\Omega_T(\lambda)u(\lambda)$ is in $H^2(\mathfrak{D}_T)$ and thus Ω_T defines a bounded operator on $H^2(\mathfrak{D}_{T^*})$.

Lemma 1. *If T is power-bounded and $\Theta_T(\lambda)$ is an invertible operator for every λ in the open unit disk, then $\sigma(T)$ is contained in the unit circle.*

Proof. Note that $\Theta_T(0)=-TJ_T|\mathfrak{D}_T$. Since $J_T|\mathfrak{D}_T$ is a symmetry it follows that T maps $\mathfrak{D}_T$ onto $\mathfrak{D}_{T^*}$. If $h\in\mathfrak{D}_{T^*}^{\perp}$ then $h=TT^*h$. Hence T is onto. Since $\Theta_{T^*}(0)=$

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$=\Theta_T^*(0)$ is invertible, the same argument proves that T^* is onto. Thus T is invertible. A simple modification of [3, p. 229, Sec. 3] shows that by taking Möbius transforms it follows that $T-\lambda$ is invertible for all λ in the open unit disk.

We are now able to state the following proposition. Though the fact seems to be known, we have not been able to find an explicit proof of it anywhere. Hence for the sake of completeness we give a brief sketch of the proof.

Proposition 1. *If T is power-bounded and $\sup_{|\lambda|<1}\|\Omega_T(\lambda)\|<\infty$ then T is similar to a unitary operator.*

Proof of this proposition rests on the result stated by Foiaş, in [2, p. 437] which we prove as a lemma.

Lemma 2. *If T is invertible then $\Theta_{T^{-1}}(\lambda^{-1})$ coincides with $\Theta_T(\lambda)$ up to constant affine factors.*

Proof. Note that $I-T^{-1*}T^{-1}=-(TT^*)^{-1}(I-TT^*)=-(I-TT^*)(TT^*)^{-1}=$ $=-(TT^*)^{-1/2}(I-TT^*)(TT^*)^{-1/2}$. Hence $\mathfrak{D}_{T^{-1}}=\mathfrak{D}_{T^*}$ and similarly $\mathfrak{D}_T=\mathfrak{D}_{T^{*-1}}$. Also $T^{-1}\mathfrak{D}_{T^*}=\mathfrak{D}_{T^{-1*}}$ and $T^*\mathfrak{D}_{T^{-1}}=\mathfrak{D}_T$. If $S=(TT^*)^{-1/2}$ then $\|Q_{T^{-1}}h\|^2=$ $=\|Q_{T^*}Sh\|^2$ and hence there exists a unitary operator $Z:\mathfrak{D}_{T^{-1}}\to\mathfrak{D}_{T^*}$ such that $ZQ_{T^{-1}}=Q_{T^*}S=SQ_{T^*}$. Now

$$\Theta_T(\xi)Q_TJ_T|\mathfrak{D}_T = Q_{T^*}(I-\xi T^*)^{-1}(\xi-T)|\mathfrak{D}_T \quad \text{[see 3, p. 227]}$$

and

$$\Theta_{T^{-1}}(\xi^{-1})Q_{T^{-1}}J_{T^{-1}}|\mathfrak{D}_{T^{-1}} = Q_{T^{*-1}}(I-\xi^{-1}T^{*-1})^{-1}(\xi^{-1}-T^{-1})|\mathfrak{D}_{T^{-1}} =$$

$$= Q_{T^{*-1}}T^*(I-\xi T^*)^{-1}(\xi-T)T^{-1}|\mathfrak{D}_{T^{-1}} = T^*Q_{T^{-1}}(I-\xi T^*)^{-1}(\xi-T)T^{-1}|\mathfrak{D}_{T^{-1}} =$$

$$= T^*Z^{-1}SQ_{T^*}(I-\xi T^*)^{-1}(\xi-T)T^{-1}|\mathfrak{D}_{T^{-1}} = T^*Z^{-1}S\Theta_T(\xi)Q_TJ_TT^{-1}|\mathfrak{D}_{T^{-1}} =$$

$$= T^*Z^{-1}S\Theta_T(\xi)Q_TJ_TT^{-1}|\mathfrak{D}_{T^{-1}} = T^*Z^{-1}S\Theta_T(\xi)J_TT^{-1}Q_{T^*}|\mathfrak{D}_{T^{-1}}.$$

Hence

$$\Theta_{T^{-1}}(\xi^{-1})J_{T^{-1}}Z^{-1}SQ_{T^*} = T^*Z^{-1}S\Theta_T(\xi)J_TT^{-1}Q_{T^*}$$

and thus

$$\Theta_{T^{-1}}(\xi^{-1})J_{T^{-1}}Z^{-1}S|\mathfrak{D}_{T^*} = T^*Z^{-1}S\Theta_T(\xi)J_TT^{-1}|\mathfrak{D}_{T^{-1}}$$

which gives the result.

Proof of proposition 1. By lemma 1, $\Theta_T(\xi)$ is defined for all ξ off the unit circle and $\Theta_T^{-1}(\xi)=J_T\Theta_T(\bar{\xi}^{-1})^*J_{T^*}|\mathfrak{D}_{T^*}$ by [1, p. 129]. Hence there exist bounded operators X and Y such that for $|\xi|<1$, $\Theta_{T^{-1}}(\xi)=X\Omega_T(\bar{\xi})^*Y$. Now if $\sup_{|\xi|<1}\|\Omega_T(\xi)\|<\infty$ then by the main theorem in [1, p. 127] T^{-1} is similar to a contraction. The result follows from the well-known theorem of Sz.-Nagy [4].

Lemma 3. $(J_T\Omega_T(\lambda)Q_{T^*}h,\ \Omega_T(\lambda)Q_{T^*}h')=(1-|\lambda|^2)((\lambda-T)^{-1}Q_{T^*}^2J_{T^*}h,\ (\lambda-T)^{-1}Q_{T^*}^2J_{T^*}h')+(Q_{T^*}^2J_{T^*}h,\ h')$ *for all $|\lambda|<1$ and all h, h' in $\mathfrak{H}$.*

Proof. It follows from Lemma 1 that $\Theta_T(\lambda)$ is defined off the unit circle. Hence by [1, p. 129],

$$\Omega_T(\lambda) = [-T^* J_{T^*} + J_T Q_T (\lambda - T)^{-1} Q_{T^*} J_{T^*}] | \mathfrak{D}_{T^*}$$

and

$$\Omega_T(\lambda) Q_{T^*} = J_T Q_T (\lambda - T)^{-1} (I - \lambda T^*).$$

Since $J_T Q_T^2 = Q_T^2 J_T = I - T^* T$ the argument in [3, chap. 6, sec. 4] gives the required relation.

Corollary 1. *For all h in $\mathfrak{H}$,* $(J_T \omega_0 Q_{T^*} h, \omega_0 Q_{T^*} h) = \|h\|^2 - \|T^* h\|^2 + \|T^{-1} Q_{T^*}^2 J_{T^*} h\|^2$.

Corollary 2. *If* $\sup_{|\lambda|<1} \|\Omega_T(\lambda)\| < \infty$ *and* $m(\sigma(T)) = 0$ *then for almost all t and all h, h' in $\mathfrak{H}$ we have*

$$\big(J_T \Omega_T(e^{it}) Q_{T^*} h, \Omega_T(e^{it}) \Omega_{T^*} h'\big) = \big((I - TT^*) h, h'\big).$$

Proof. If $e^{it} \notin \sigma(T)$ then $\|(re^{it} - T)^{-1}\|$ is bounded in a neighborhood of e^{it} and as $r \to 1-$, $(1 - r^2)\|(re^{it} - T)^{-1}\| \to 0$.

Proposition 2. *If* $\sup_{|\lambda|<1} \|\Omega_T(\lambda)\| < \infty$ *and* $m(\sigma(T)) = 0$ *then T is a unitary operator.*

It is convenient to break up the proof in two steps, formulated as Lemma 4 and 5.

Lemma 4. *For all h in $\mathfrak{H}$,* $\sum_{n=1}^{\infty} (J_T \omega_n Q_{T^*} h, \omega_n Q_{T^*} h) = -\|T^{-1} Q_{T^*}^2 J_{T^*} h\|^2$.

Proof. For h in $\mathfrak{H}$, $Q_{T^*} h$ can be considered as a constant function in $H^2(\mathfrak{D}_{T^*})$. If $(Ju)(\lambda) = J_T(u(\lambda))$ for u in $H^2(\mathfrak{D}_T)$ and $|\lambda| < 1$ than $H^2(\mathfrak{D}_T)$ becomes a J-space and we have

$$(J\Omega_T(Q_{T^*} h), \Omega_T(Q_{T^*} h)) = \frac{1}{2\pi} \int_0^{2\pi} (J_T \Omega_T(e^{it}) Q_{T^*} h, \Omega_T(e^{it}) Q_{T^*} h)\, dt = \|h\|^2 - \|T^* h\|^2$$

by Corollary 2. On the other hand,

$$\Omega_T(Q_{T^*} h) = \sum_{n=0}^{\infty} \lambda^n \omega_n Q_{T^*} h \quad \text{and} \quad J\Omega_T(Q_{T^*} h) = \sum_{n=0}^{\infty} \lambda^n J_T \omega_n Q_{T^*} h.$$

Thus

$$\|h\|^2 - \|T^* h\|^2 = \sum_{n=0}^{\infty} (J_T \omega_n Q_{T^*} h, \omega_n Q_{T^*} h).$$

Applying Corollary 1 we get the result.

Lemma 5. *For all h in $\mathfrak{H}$,* $\lim_{n\to\infty} \|T^{-n} T^* h - T^{-(n+1)} h\| = 0$ *and hence* $T^* h = T^{-1} h$.

Proof. It follows as in the proof of Lemma 3 that

$$\omega_0 = -[T^* J_{T^*} + Q_T J_T T^{-1} Q_{T^*} J_{T^*}] | \mathfrak{D}_{T^*}$$

and

$$\omega_n = -Q_T J_T T^{-(n+1)} Q_{T^*} J_{T^*} | \mathfrak{D}_{T^*}$$

for $n \geqq 1$.

Hence for $n \geqq 1$,

$$(J_T \omega_n Q_{T^*} h, \omega_n Q_{T^*} h) = ((I - T^* T)[T^{-n} T^* h - T^{-(n+1)} h], [T^{-n} T^* h - T^{-(n+1)} h]) =$$
$$= \|T^{-n} T^* h - T^{-(n+1)} h\|^2 - \|T^{-(n-1)} T^* h - T^{-n} h\|^2.$$

Thus

$$\sum_{n=1}^{\infty} (J_T \omega_n Q_{T^*} h, \omega_n Q_{T^*} h) = \lim_{m \to \infty} \sum_{n=1}^{m} (J_T \omega_n Q_{T^*} h, \omega_n Q_{T^*} h) =$$
$$= \lim_{m \to \infty} \|T^{-m} T^* h - T^{-(m+1)} h\|^2 - \|T^* h - T^{-1} h\|^2.$$

Since $\|T^{-1} Q_{T^*}^2 J_{T^*} h\|^2 = \|T^{-1}(I - TT^*) h\|^2 = \|T^{-1} h - T^* h\|^2$ an application of Lemma 4 gives that $\lim_{n \to \infty} \|T^{-n}(T^* h - T^{-1} h)\| = 0$. Since T is power-bounded we have $T^* h = T^{-1} h$.

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