Invariant subspaces of von Neumann algebras

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In what follows we denote by H a complex Hilbert space and by $B(H)$ the algebra of all bounded linear operators on H. A vector subspace $K \subset H$ is called semi-closed if there is $t \in B(H)$ such that $K = tH$. An operator $T\colon D_T \to H$ ($D_T \subset H$) is called semi-closed if its graph $\Gamma_T = \{(x, Tx) | x \in D_T\}$ is a semi-closed subspace of $H \oplus H$. If $B \subset B(H)$, we shall denote by $\mathrm{Lat}\,(B)$ the set of all closed subspaces of H, invariant for B and by $\mathrm{Lat}_{1/2}\,(B)$ the set of all semi-closed subspaces of H invariant for B. For $n \in \mathrm{N}$, we denote

$$H^{(n)} = \underbrace{H \oplus H \oplus \ldots \oplus H}_{n \text{ fold}} \quad \text{and} \quad B^{(n)} = \{\underbrace{a \oplus a \oplus \ldots \oplus a}_{n \text{ fold}} | a \in B\}.$$

We say that an algebra $A \subset B(H)$ is *transitive* if it is weakly closed in $B(H)$ and $\mathrm{Lat}\,(A) = \{(0), H\}$. In [1], [2] (see [8] p. 138) there are given conditions for a transitive algebra to be equal to $B(H)$. An algebra $A \subset B(H)$ is called *strongly transitive* if it is weakly closed in $B(H)$ and $\mathrm{Lat}_{1/2}\,(A) = \{(0), H\}$.

In [3], C. FOIAŞ has proved that the only strongly transitive algebra is $B(H)$. We say that an algebra $A \subset B(H)$ is reductive if it is weakly closed and $\mathrm{Lat}\,(A) = \mathrm{Lat}\,(A^*)$ (where $A^* = \{a^* | a \in A\}$).

In [4], [7] (see [8], p. 167) there are given conditions for a reductive algebra to be a von Neumann algebra. Finally, an algebra $A \subset B(H)$ is called *strongly reductive* if it is weakly closed and $\mathrm{Lat}_{1/2}(A^*) \subset \mathrm{Lat}_{1/2}(A)$. In [9], D. VOICULESCU has proved that if A is a weakly closed algebra with spatial multiplicity $\geqq 3$ and such that $\mathrm{Lat}_{1/2}\,(A) = \mathrm{Lat}_{1/2}\,(M)$, where M is the von Neumann algebra generated by A (in particular A is strongly reductive), then $A = M$. Our corollary 1.3 generalizes this result. In § 2 we study reductive algebras which contain von Neumann algebras having property (P) of J. T. SCHWARTZ.

Recall that a von Neumann algebra N has property (P), if for every $t \in B(H)$ the weakly closed convex hull of $\{u^* t u | u \in N, \text{ unitary}\}$ has non-void intersection with the commutant N' of N.

§ 1. Strongly reductive algebras

1.1. Lemma. (See [7]). *Let A and M be weakly closed algebras such that $A \subset M$ and* Lat $(A^{(n)}) =$ Lat $(M^{(n)})$ *for every* $n \in \mathrm{N}$. Then $A = M$.

The following theorem appears in literature in an implicite form:

1.2. Theorem. *Let $A \subset B(H)$ be a reductive algebra. We suppose that for any finite collection $T_1, \ldots, T_n$ of linear oparetors defined on one and the same dense subspace $K \subset H$, the relation $K_{n+1} = \{(x_1, T_1 x, \ldots, T_n x) | x \in K\} \in$ Lat $(A^{(n+1)})$ implies that $K_{n+1} \in$ Lat $(A^{*(n+1)})$. Then A is a von Neumann algebra.*

Proof. We shall prove by induction that the assumption of Lemma 1.1 is also satisfied if M is replaced by the von Neumann algebra M which A generates. In fact, by the reductivity of A we have Lat $(A) =$ Lat (M). Suppose that for $k \leqq n$, Lat $(A^{(k)}) =$ Lat $(M^{(k)})$ and let $L_{n+1} \in$ Lat $(A^{(n+1)})$. Set $L_n = \{(x_1, \ldots, x_{n+1}) \in L_{n+1} | x_1 = 0\}$. As L_n can be considered an element of Lat $(A^{(n)})$ the induction hypothesis implies that $L_n \in$ Lat $(M^{(n+1)})$. Since M is a von Neumann algebra, we have $H^{(n+1)} \ominus L_n \in$ $\in$ Lat $(M^{(n+1)}) \subset$ Lat $(A^{(n+1)})$. Therefore $L_{n+1} \ominus L_n = (H^{(n+1)} \ominus L_n) \cap L_{n+1} \in$ Lat $(A^{(n+1)})$. If $(x_1, \ldots, x_{n+1}) \in L_{n+1} \ominus L_n$ and $x_1 = 0$, then $x_2 = \ldots = x_{n+1} = 0$.

It follows that there exists a linear subspace $K_0 \subset H$ and linear operators $T_1^0, \ldots,$ T_n^0 defined on K_0 such that $L_{n+1} \ominus L_n = \{(x, T_1^0 x \ldots T_1^0 x) | x \in K_0\}$.

For every i $(1 \leqq i \leqq n)$ we define on the dense subspace $K = K_0 + K_0^{\perp}$ the operator T_i in the following way:

$$T_i x = T_i^0 x \quad \text{if} \quad x \in K_i, \quad T_i x = 0 \quad \text{if} \quad x \in K_0^{\perp}$$

It is obvious that

$$L_{n+1} \ominus L_n = \{(x, T_1 x, \ldots, T_n x) | x \in K\} \ominus (K_0 \oplus (0) \oplus \ldots \oplus (0))$$

and that $\{x, T_1 x, \ldots, T_n x) | x \in K\} \in$ Lat $(A^{(n+1)})$. By the assumption of the theorem, $\{(x, T_1 x, \ldots, T_n x) | x \in K\} \in$ Lat $(M^{(n+1)})$, and by the reductivity of A, we have $K_0 \oplus$ $\oplus (0) \ldots \oplus (0) \in$ Lat $(M^{(n+1)})$. It follows that $L_{n+1} \ominus L_n \in$ Lat $M^{(n+1)}$. Therefore, $L_{n+1} =$ $= (L_{n+1} \ominus L_n) \oplus L_n \in$ Lat $M^{(n+1)}$.

1.3. Corollary. *Let $A \subset B(H)$ be an algebra such that $A^{(2)}$ is strongly reductive Then A is a von Neumann algebra.*

Proof. Let $K \subset H$ be a dense subspace and $T_i \colon K \to H$ $(i = 1, \ldots, n)$ be linear operators such that $K_{n+1} = \{(x, T_1 x, \ldots, T_n x) | x \in K\} \in$ Lat $(A^{(n+1)})$. It is obvious that each T_i $(1 \leqq i \leqq n)$ commutes with A on K.

Let p_{1i} be the projection of $H^{(n+1)}$ onto the first and i th component $(i = 1, \ldots, n)$. Then

$$p_{1i} K_{n+1} = \{(x, T_i x) | x \in K\} \in \mathrm{Lat}_{1/2}(A^{(2)}) \subset \mathrm{Lat}_{1/2}(A^{*(2)}).$$

Therefore each T_i $(i=1, \ldots, n)$ commutes with A^* on K. It follows that $K_{n+1} \in \in \mathrm{Lat}\,(A^{*(n+1)})$.

By Theorem 1.2 it follows that A is a von Neumann algebra.

§ 2. Reductive algebras

In [1] it is shown that if a reductive algebra A contains a m.a.s.a (maximal abelian self adjoint algebra), then A is a von Neumann algebra. In [2], a more general result is proved: if a reductive algebra A contains an abelian von Neumann algebra with finite commutant, then A is a von Neumann algebra. It is known that the commutative von Neumann algebras (and more generally type I von Neumann algebras) have property (P).

Taking into account Theorem 2.2 below, it is likely that the answer to the following question is in the affirmative:

2.1. Question. If A is a reductive algebra which contains a von Neumann algebra N with property (P) and having finite commutant, then A is a von Neumann algebra.

A partial answer to this question is given by

2.2. Theorem. *Let $A \subset B(H)$ be an algebra such that* 1) *$A^{(2)}$ is reductive;* 2) *$A^{(2)}$ contains a von Neumann algebra $N^{(2)}$ with property (P) and having finite commutant. Then A is a von Neumann algebra.*

In the proof of this theorem we need the following:

2.3. Lemma. *Let $N \subset B(H)$ be a von Neumann algebra with finite commutant. If $N^{(2)}$ has property (P), then every semi-closed, densely defined operator which commutes with N is preclosed.*

Proof. Let $T\colon D_T \to H$ be a semi-closed, densely defined linear operator which commutes with N. Then the linear subspace $\Gamma_T = \{(x, Tx) | x \in D_T\} \subset H^{(2)}$ is a semi-closed subspace, invariant under $N^{(2)}$. Because $N^{(2)}$ has property (P), it follows (cf. [9], Théorème 2) that there exists an operator $Q \in N^{(2)\prime}$ such that $\Gamma_T = Q(H^{(2)}) = Q((\ker Q)^\perp)$. Hence for each $x \in D_T$ there exists a unique $(y_1(x), y_2(x)) \in (\ker Q)^\perp$ such that $(x, Tx) = Q(y_1(x), y_2(x))$. Set $\Delta = \{(x, x) \in H^{(2)} | x \in H\}$.

We now define a linear operator Y on the dense linear subspace $D_Y = (\Delta \cap D_T^{(2)}) + \Delta^\perp \subset H^{(2)}$ as follows:

$$Y(x, x) = (y_1(x), y_2(x)) \quad \text{for} \quad x \in D_T; \qquad Y(z, y) = 0 \quad \text{for} \quad (z, y) \in \Delta^\perp$$

The operator Y is closed. Indeed, let $\{(x_n, x_n) + (z_n, y_n)\}_{n \in \mathbf{N}}$ be such that $(x_n, x_n) + (z_n, y_n) \to (x, x) + (z, y)$ $(x \in H, (z, y) \in \Delta^\perp)$ and $Y((x_n, x_n) + (z_n, y_n)) = (y_1(x_n), y_2(x_n)) \to (u, v) \in (\ker Q)^\perp$ as $n \to \infty$.

Because of the continuity of Q, it follows that $Q(y_1(x_n), y_2(x_n)) \to Q(u, v)$. Therefore $(x_n, Tx_n) \to Q(u, v)$ and $Q(u, v) = Q(y_1(x), y_2(x))$. It follows that $(u, v) = (y_1(x), y_2(x))$ and hence Y is closed. We will show that Y commutes with $N^{(2)}$. Since $Q \in N^{(2)'}$ we obtain that $(\ker Q)^\perp$ is invariant under $N^{(2)}$. Now for $x \in D_T$ and $a \in N$ we have

$$a^{(2)}(x, Tx) = (ax, Tax) = Q(y_1(ax), y_2(ax)).$$

On the other hand:

$$a^{(2)}(x, Tx) = a^{(2)} Q(y_1(x), y_2(x)) = Q(ay_1(x), ay_2(x)).$$

By the remark above $(ay_1(x), ay_2(x)) \in (\ker Q)^\perp$, and therefore $(y_1(ax), y_2(ax)) = (ay_1(x), ay_2(x))$.

Since Δ is an invariant subspace under $N^{(2)}$ and $N^{(2)}$ is a von Neumann algebra, it follows that $\Delta^\perp$ is invariant under $N^{(2)}$. Therefore Y commutes with $N^{(2)}$. Let p_2 be the projection of $H^{(2)}$ onto its 2nd component. It is obvious that $Tx = p_2 Q Y(x, x)$. Since $p_2 Q \in N^{(2)'}$ and Y is affiliated to $N^{(2)}$ (which is a finite von Neumann algebra), we obtain (cf. [5] and also [6], Theorem XV, p. 119) that $p_2 Q Y$ is preclosed and therefore T is preclosed.

Proof of Theorem 2.2. We shall verify the hypothesis of Theorem 1.2. Let $K \subset H$ be a dense subspace, and $T_1, \dots, T_n$ linear operators defined on K and such that $K_{n+1} = \{(x, T_1 x, \dots, T_n x) | x \in K\} \in \mathrm{Lat}\,(A^{(n+1)})$. As in the proof of Corollary 1.3, it follows that for every i $(1 \leqq i \leqq n)$ the graph $\Gamma_{T_i} = \{(x, T_i x) | x \in K\}$ is semi-closed and therefore the operators T_i, $1 \leqq i \leqq n$, are semi-closed.

By Lemma 2.3 the operators T_i $(1 \leqq i \leqq n)$ are preclosed. Let $\overline{T}_i$ be the closure of T_i $(1 \leqq i \leqq n)$, and $K_0 = \bigcap_{i=1}^{n} D_{\overline{T}_i}$. Obviously, $K \subset K_0$. Since $A^{(2)}$ is reductive, $\overline{T}_i$ commutes with A^*. Set $\Delta_n = \{(x, x, \dots, x) \in H^{(n)} | x \in H\}$ and define the operators T and T_0 on the dense subspaces $(\Delta_n \cap K^{(n)}) + \Delta_n^\perp \subset H^{(n)}$ and $\Delta_n K^{(n)} + \Delta_n^\perp \subset H^{(n)}$ respectively in the following way:

$$T(x, x, \dots, x) = (T_1 x, \dots, T_n x) \quad \text{if} \quad (x, \dots, x) \in \Delta_n \cap K^{(n)},$$

$$T(x_1, \dots, x_n) = 0 \quad \text{if} \quad (x_1, \dots, x_n) \in \Delta_n^\perp$$

and

$$T_0(x, x, \dots, x) = (\overline{T}_1 x, \dots, \overline{T}_n x) \quad \text{if} \quad (x, \dots, x) \in \Delta_n \cap K_0^{(n)},$$

$$T_0(x_1, \dots, x_n) = 0 \quad \text{if} \quad (x_1, \dots, x_n) \in \Delta_n^\perp.$$

Because Δ_n is invariant under $A^{(n)}$, and therefore under $N^{(n)}$, it is easily seen that T and T_0 are closed operators affiliated with the finite von Neumann algebra

$N^{(n)}$. Thus $K \subset K_0$ implies that $T \subset T_0$. According to [5] (see also [6], Theorem XV, p. 119) we obtain that $T = T_0$. By the remark above T_0 commutes with $A^{*(n)}$, and therefore T commutes with $A^{*(n)}$. But this means that $K_{n+1} \in \mathrm{Lat}\,(A^{*(n+1)})$.

Added in proof. We remark that Lemma 2.3. holds without the assumption „$N^{(2)}$ has property (P)", so Theorem 2.2. can be improved: Let $A \subset B(H)$ be an algebra which contains a von Neumann algebra with finite commutant and such that $A^{(2)}$ is reductive. Then A is a von Neumann algebra. Proofs of these improvements will appear elsewhere.

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