On orthogonal trigonometric polynomials

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Dedicated to Professor K. Tandori on his 50th birthday

1. Let Φ, $\mathcal{T}$, $\mathcal{P}$ respectively denote the set of all orthonormal systems $\varphi = \{\varphi_n(x)\}_1^\infty$, the set of orthonormal systems $T = \{T_n(x)\}_1^\infty$ consisting of trigonometric polynomials, and the set of orthonormal systems $P = \{P_n(x)\}_1^\infty$ consisting of algebraic polynomials, on the interval $[0, 2\pi]$.

For any given set $\mathcal{H}$ of orthonormal systems $H = \{H_n(x)\}_1^\infty$ on $[0, 2\pi]$, a sequence $\{a_n\}_1^\infty$ of real numbers is said to be a *convergence sequence over* $\mathcal{H}$ if for each $H \in \mathcal{H}$ the series $\sum_{n=1}^{\infty} a_n H_n(x)$ converges almost everywhere in $[0, 2\pi]$.

For any sequence $\{a_n\}_1^\infty$ of real numbers we define

$$\|\{a_n\}_M^N, \mathcal{H}\|_p = \sup_{H \in \mathcal{H}} \Big(\int_0^{2\pi} \sup_{M \leq i < j < N} \Big|\sum_{n=i+1}^{j} a_n H_n(x)\Big|^p dx\Big)^{1/p}$$

$(1 \leq p \leq 2;\ 0 \leq M < N \leq \infty)$.

It can be shown that

$$\lim_{N \to \infty} \|\{a_n\}_0^N, \mathcal{H}\|_p = \|\{a_n\}_0^\infty, \mathcal{H}\|_p. \tag{1}$$

In [3] TANDORI proved the following

Theorem A. *The sequence $\{a_n\}_1^\infty$ is a convergence sequence over Φ if and only if $\|\{a_n\}_1^\infty, \Phi\|_p < \infty$ $(1 \leq p \leq 2)$.*

In [1] LEINDLER proved two deep approximation theorems for orthonormal polynomials and using these he proved, roughly saying, that if a divergence theorem can be sated for a general orthogonal series then there exists a series of orthogonal polynomials for which the same divergence phenomenon holds.

In the present paper we prove the analogues of Leindler's theorems for orthogonal trigonometric polynomials.

Theorem 1. Let $\varphi \in \Phi$. *For any sequence $\{\varepsilon_k\}_1^\infty$ of positive numbers and any sequence $\{N_k\}_0^\infty$ of integers $(0 = N_0 < N_1 < \ldots)$ there exist a system $T \in \mathcal{T}$ and a sequence*

$\{G_k\}_1^\infty$ of measurable subsets of $[0, 2\pi]$ such that for any $x \in CG_k$ and n satisfying $N_{k-1} < n \leqq N_k$ we have

$$|\varphi_n(x) - (-1)^{j_k(x)} T_n(x)| \leqq \varepsilon_k \qquad (j_k(x) = 0 \text{ or } 1), \tag{2}$$

$$\mu(G_k) \leqq \varepsilon_k \qquad (k = 1, 2, \ldots), \tag{3}$$

and

$$\max_{x \in [0, 2\pi]} |T_n(x)| \leqq \sqrt{2}\Big(\sup_{0 < x < 2\pi} |\varphi_n(x)| + 1\Big). \tag{4}$$

Theorem 2. *Let $\varphi \in \Phi$. Let $\{a_n\}_1^\infty$ be a sequence of real numbers and $\{b_n\}_1^\infty$ a non-decreasing sequence of positive numbers. Suppose that $\{\mathscr{H}_k\}_1^\infty$ is a sequence of measurable subsets of $[0, 2\pi]$, $\{N_k\}_0^\infty$ is a given sequence of integers $(0 = N_0 < N_1 < \ldots)$, and ε is a given positive number. If $\mu(\overline{\lim_k} \mathscr{H}_k) = 2\pi$ if and for each $x \in \mathscr{H}_k$ there is a pair of integers $\nu_k(x)$, $\mu_k(x)$ such that $N_k \leqq \nu_k(x) < \mu_k(x) \leqq N_{k+1}$ and*

$$\left| \sum_{n_k = \nu(x)+1}^{\mu_k(x)} a_n \varphi_n(x) \right| \geqq b_k, \tag{5}$$

then there exists a $T \in \mathscr{T}$ such that the inequality

$$\left| \sum_{n = \nu_k(x)+1}^{\mu_k(x)} a_n T_n(x) \right| \geqq (1 - \varepsilon) b_k \tag{6}$$

holds for infinitely many k almost everywhere in $[0, 2\pi]$. If the system φ is uniformly bounded then the system T can also be chosen uniformly bounded.

Using Theorems 1 and 2 and results of TANDORI we prove the following theorems.

Theorem 3. *If $1 \leqq p \leqq 2$ then the inequalities*

$$\|\{a_n\}_0^\infty, \mathscr{T}\|_p \leqq \|\{a_n\}\|_0^\infty, \Phi\|_p \leqq 2^{1-\frac{1}{p}} \|\{a_n\}_0^\infty, \mathscr{T}\|_p \tag{7}$$

hold.

Theorem 4. *The sequence $\{a_n\}_1^\infty$ is a convergence sequence over $\mathscr{T}$ if and only if $\|\{a_n\}_0^\infty, \mathscr{T}\|_p < \infty$ $(1 \leqq p \leqq 2)$.*

Finally, from Theorem A and Theorems 3, 4 we get immediately

Theorem 5. *A sequence $\{a_n\}_1^\infty$ of reals is a convergence sequence over Φ if and only if it is a convergence sequence over $\mathscr{T}$.*

We remark that Theorems 3—5 hold true for $\mathscr{P}$ instead of $\mathscr{T}$, too.

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2. We require the following lemmas. The proof of our first lemma is completely similar to that of one of LEINDLER's lemmas ([1], p. 26) so we omit its proof.

Lemma 1. *Let* $\{\psi_n(x)\}_1^\infty$ *be a system of measurable and bounded functions, and* $\{N_k\}_0^\infty$ *a given sequence of integers* $(0=N_0<N_1<\dots)$. *If for each* k $(k=1,2,\dots)$ *the system* $\{\psi_n(x)\}_{N_{k-1}+1}^{N_k}$ *is orthonormal in the interval* $[0,2\pi]$ *then for every given sequence* $\{\varepsilon_k\}_1^\infty$ *of positive numbers there exist a system* $T\in\mathcal{T}$ *and a sequence* $\{E_k\}_1^\infty$ *measurable subsets of* $[0,2\pi]$ *such that for any* $x\in\complement E_k$ *and* $N_{k-1}<n\leqq N_k$

(8) $$|\psi_n(x)-(-1)^{j_k(x)}|T_n(x)<\varepsilon_k \qquad (j_k(x)=0 \text{ or } 1),$$

(9) $$\mu(E_k)\leqq\varepsilon_k \qquad (k=1,2,\dots)$$

and

(10) $$\max_{x\in[0,2\pi]}|T_n(x)|\leqq\sqrt{2}\big(\sup_{0<x<2\pi}|\psi_n(x)|+1\big) \qquad (n=1,2,\dots).$$

Lemma 2. (LEINDLER [1], p. 33) *Let* $\varphi\in\Phi$. *For every given sequence* $\{\varepsilon_k\}_1^\infty$ *of positive numbers and any sequence* $\{N_k\}_0^\infty$ *of integers* $(0=N_0<N_1<\dots)$, *there exist a normed system* $\{\psi_n(x)\}_1^\infty$ *of measurable and bounded functions and a sequence* $\{\mathcal{H}_k\}_1^\infty$ *of measurable subsets of* $[0,2\pi]$ *such that, for every* k $(k=1,2,\dots)$,

(11) $$\int_0^{2\pi}\psi_n(x)\psi_m(x)\,dx=0 \qquad (N_{k-1}<n<m\leqq N_k),$$

(12) $$|\varphi_n(x)-\psi_n(x)|<\varepsilon_k \quad \textit{on} \quad \complement\mathcal{H}_k \qquad (N_{k-1}<n\leqq N_k),$$

(13) $$\mu(\mathcal{H}_k)\leqq\varepsilon_k,$$

(14) $$\sup_{0<x<2\pi}|\psi_n(x)|\leqq\sup_{0<x<2\pi}|\varphi_n(x)|.$$

On the basis of a lemma of TANDORI [3], p. 222, and by (1) we get

Lemma 3. *If* $1\leqq p\leqq 2$ *and* $1\leqq N\leqq\infty$ *then*

$$\varrho\|\{a_n\}_0^N,\Phi\|_2\leqq\|\{a_n\}_0^N,\Phi\|_p\leqq\|\{a_n\}_0^N,\Phi\|_2$$

where ϱ *is a positive absolute constant.*

Lemma 4. (TANDORI [3], p. 220) *If* $1\leqq M<N<\infty$ *then*

$$\|\{a_n\}_0^N,\Phi\|_2\leqq\|\{a_n\}_0^{M+1},\Phi\|_2+\|\{a_n\}_M^N,\Phi\|_2.$$

A partial result in the proof of TANDORI's theorem ([2], p. 146) we use as

Lemma 5. *Let* $\{N_k\}_0^\infty$ *be a given sequence of integers* $(0=N_0<N_1<\dots)$. *If*

(15) $$\sum_{k=0}^\infty\|\{a_n\}_{N_k}^{N_{k+1}+1},\Phi\|_2^2=\infty,$$

then there exist a system $\varphi\in\Phi$ *and a sequence* $\{E_k\}_1^\infty$ *of stochastically independent subsets of* $[0,2\pi]$ *(every* E_k *is a union of intervals of finite number) such that for each* k

(16) $$\mu(E_k)\geqq\alpha\|\{a_n\}_{N_k}^{N_{k+1}+1},\Phi\|_2^2 \quad (\alpha \textit{ is a positive constant}),$$

furthermore there exist integers $\nu_k=\nu_k(x)$, $\mu_k=\mu_k(x)$ *such that* $N_k\leqq\nu_k(x)<\mu_k(x)\leqq$

$\leqq N_{k+1}$ *and*

(17) $$\left|\sum_{n=\nu_k(x)+1}^{\mu_k(x)} a_n \varphi_n(x)\right| \geqq 1 \quad \textit{for} \quad x \in E_k.$$

3. Proof of Theorem 1. Applying Lemma 2 to the system φ and the sequences $\left\{\frac{\varepsilon_k}{2}\right\}_1^\infty$ and $\{N_k\}_0^\infty$ we get that there exist a normed system of measurable and bounded functions ψ and a sequence $\{\mathcal{H}_k\}_1^\infty$ of measurable sets such that (11) is fulfilled. By (12) and (13) we have that $\mu(\mathcal{H}_k)<\frac{\varepsilon_k}{2}$ and if $x\in \mathbf{C}\mathcal{H}_k$ then $|\varphi_n(x)-\psi_n(x)|<\frac{\varepsilon_k}{2}$ $(N_{k-1}<n\leqq N_k;\ k=1,2,\dots)$. Now applying Lemma 1 with the system ψ and the above mentioned sequences we obtain that there exist a system T and a sequence $\{E_k\}_1^\infty$ of measurable sets such that $\mu(E_k)<\frac{\varepsilon_k}{2}$ (see (9)) and if $x\in\mathbf{C}E_k$ then

$$|\psi_n(x)-(-1)^{j_k(x)}T_n(x)| < \frac{\varepsilon_k}{2} \qquad (N_{k-1} < n \leqq N_k;\ k = 1, 2, \dots;\ j_k(x) \text{ as in } (8)).$$

Let $G_k=\mathcal{H}_k\cup E_k$ $(k=1,2,\dots)$. Collecting the above facts we immediately obtain (2) and (3). By (14) and (10) we have (4), too.

4. Proof of Theorem 2. Let

(18) $$\varepsilon_k = \varepsilon/[2^k(N_k-N_{k-1})\max\{1, |a_{N_{k-1}+1}|, \dots, |a_{N_k}|\}].$$

Applying Theorem 1 to the system φ and the sequence $\{\varepsilon_k\}_1^\infty$ and $\{N_k\}_0^\infty$ we get that there exist a system T and a sequence of measurable sets $\{G_k\}_1^\infty$ such that (2) and (3) are fulfilled.

Let us choose a natural number ν such that $2^{-(\nu+1)}\leqq b_1$. If $k\geqq\nu$ and $x\in\mathcal{H}_k-G_{k+1}$ then using (2), (5) and (18) we obtain

$$b_k \leqq \left|\sum_{n=\nu_k(x)+1}^{\mu_k(x)} (-1)^{j_{k+1}(x)} a_n T_n(x)\right| + \left|\sum_{n=\nu_k(x)+1}^{\mu_k(x)} a_n\big(\varphi_n(x)-(-1)^{j_{k+1}(x)}T_n(x)\big)\right| \leqq$$

$$\leqq \left|\sum_{n=\nu_k(x)+1}^{\mu_k(x)} a_n T_n(x)\right| + \big(\mu_k(x)-\nu_k(x)\big)\varepsilon_{k+1}\max\{|a_{\nu_k}(x)+1|, \dots, |a_{\mu_k}(x)|\} \leqq$$

$$\leqq \left|\sum_{n=\nu_k(x)+1}^{\mu_k(x)} a_n T_n(x)\right| + \varepsilon b_k,$$

thus (6) holds.

It remains to show that inequality (6) is fulfilled almost everywhere in $[0, 2\pi]$, that is, to show that almost all x belong to the sets $\mathcal{H}_k-G_{k+1}$ for infinite many indexes k. Thus it is sufficient to prove that $\mu(\overline{\lim_k}\, G_k)=0$. But this follows from

$$\mu(\overline{\lim_k}\, G_k) \leqq \mu\left(\bigcup_{k=m}^{\infty} G_k\right) \leqq \sum_{k=m}^{\infty}\mu(G_k) \leqq \sum_{k=m}^{\infty}\varepsilon_k \leqq \sum_{k=m}^{\infty}(\varepsilon/2^k) = \varepsilon/2^{m-1}.$$

If the system φ is uniformly bounded, then by (4) so is the system T too.

5. Proof of Theorem 3. First of all we remark that since $\mathcal{T}\subset\Phi$, the first inequality (7) is evident. Furthermore by (1) it is enough to show that for every integer $N>0$ the inequality

$$\|\{a_n\}_0^N,\Phi\|_p \leqq 2^{1-\frac{1}{p}}\|\{a_n\}_0^N,\mathcal{T}\|_p \tag{19}$$

holds.

Let $\varphi=\{\varphi_n(x)\}_1^\infty\in\Phi$ be an arbitrary but fixed system. As the functions $\varphi_n(x)$ are square-integrable so are the function $\delta_N(x)=\max\limits_{0\leqq i<j<N}\left|\sum\limits_{n=i+1}^{j}a_n\varphi_n(x)\right|$. Therefore, for an arbitrary $\varepsilon(>0)$ there exists a $\delta'(>0)$ such that for every measurable set G with $\mu(G)<\delta'$ we have

$$\int\limits_G \delta_N^p(x)\,dx \leqq (\varepsilon/2)^p \qquad (1\leqq p\leqq 2). \tag{20}$$

For any i and j $(0\leqq i<j<N)$ let

$$\delta=\delta(i,j,N,\delta',\varepsilon,\{a_n\})=\min\{\delta'/2^{i+j},\ \varepsilon/(8N\pi\max_{1\leqq n<N}|a_n|)\}. \tag{21}$$

By Theorem 1 there exist a system $\{T_n^{(i,j)}(x)\}_1^\infty\in\mathcal{T}$ and a measurable set $G^{(i,j)}$ such that if $i<n\leqq j$ then for any $x\in\complement G^{(i,j)}$

$$|\varphi_n(x)-(-1)^{j(x)}T_n^{(i,j)}(x)|\leqq\delta \qquad (j(x)=0 \text{ or } 1) \tag{22}$$

and

$$\mu(G^{(i,j)})\leqq\delta. \tag{23}$$

If $x\in\complement G^{(i,j)}$ then by (22) we get

$$\left|\sum_{n=i+1}^{j}a_n\varphi_n(x)\right|\leqq\left|\sum_{n=i+1}^{j}a_nT_n^{(i,j)}(x)\right|+\delta\sum_{n=i+1}^{j}|a_n|$$

and considering (21) we have

$$\left|\sum_{n=i+1}^{j}a_n\varphi_n(x)\right|^p\leqq 2^{p-1}\left|\sum_{n=i+1}^{j}a_nT_n^{(i,j)}(x)\right|^p+(\varepsilon/4\pi)^p, \tag{24}$$

where $1\leqq p\leqq 2$.

Set $G_N=\bigcup\limits_{0\leqq i<j<N}G^{(i,j)}$. Using (21) and (23) we get

$$\mu(G_N)\leqq\sum_{0\leqq i<j<N}\mu(G^{(i,j)})\leqq\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\delta'/2^{i+j}=\delta'. \tag{25}$$

If $x\in\complement G_N$, by (24), we have

$$\delta_N^p(x)\leqq 2^{p-1}\max_{0\leqq i<j<N}\left|\sum_{n=i+1}^{j}a_nT_n^{(i,j)}(x)\right|^p+(\varepsilon/4\pi)^p \qquad (1\leqq p\leqq 2)$$

and considering (20) and (25) we get

$$\int_0^{2\pi} \delta_N^p(x)\,dx = \Big(\int_{CG_N} + \int_{G_N}\Big)\delta_N^p(x)\,dx \leqq$$

$$\leqq 2^{p-1}\int_0^{2\pi} \max_{0\leqq i<j<N}\left|\sum_{n=i+1}^{j} a_n T_n^{(i,j)}(x)\right|^p dx + \varepsilon^p \leqq 2^{p-1}\|\{a_n\}_0^N, \mathscr{T}\|_p^p + \varepsilon^p$$

$$(1 \leqq p \leqq 2).$$

Hence we can see that

$$\sup_{\varphi\in\Phi}\left(\int_0^{2\pi} \max_{0\leqq i<j<N}\left|\sum_{n=i+1}^{j} a_n\varphi_n(x)\right|^p dx\right)^{1/p} \leqq 2^{1-\frac{1}{p}}\|\{a_n\}_0^N, \mathscr{T}\|_p + \varepsilon.$$

Considering that ε was arbitrary small we have (19), thus our proof is complete.

6. Proof of Theorem 4. By Theorem A and Theorem 3 the sufficiency is obvious.

To prove the necessity we assume $\|\{a_n\}_0^\infty, \mathscr{T}\|_p = \infty$. Applying Theorem 3 and Lemma 3 we have $\|\{a_n\}_0^\infty, \Phi\|_2 = \infty$.

By (1) and Lemma 4 we obtain that $\lim\limits_{N\to\infty} \|\{a_n\}_M^N, \Phi\|_2 = \infty$ for any M; thus there exists a sequence $\{N_k\}_0^\infty$ $(0 = N_0 < N_1 < \dots)$ such that $\|\{a_n\}_{N_k}^{N_{k+1}+1}, \Phi\|_2 \geqq 1$ for every k.

For the sequence $\{N_k\}_0^\infty$ we can apply Lemma 5 and we get a system $\varphi\in\Phi$ and a sequence $\{E_k\}_1^\infty$ of stochastically independent sets such that (16) is fulfilled and if $x\in E_k$ then (17) holds.

Considering (15), (16), and applying the second Borel-Cantelli lemma we get $\mu(\overline{\lim_k}\, E_k) = 2\pi$.

Taking the system φ, the sequences $\{a_n\}_1^\infty$, $b_n \equiv 1$ $(n = 1, 2, \dots)$, and choosing $\varepsilon = 1/2$, it follows from Theorem 2 that there exists a system $T\in\mathscr{T}$ such that the inequality $\left|\sum_{n=\nu_k(x)+1}^{\mu_k(x)} a_n T_n(x)\right| \geqq \frac{1}{2}$ holds for infinitely many k, almost everywhere in $[0, 2\pi]$. This implies that the series $\sum_{n=1}^{\infty} a_n T_n(x)$ diverges almost everywhere in $[0, 2\pi]$.

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