On non-localizable measure spaces

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0. Introduction

In [S] I. E. SEGAL writes that "it is easily seen that a localizable space is strongly equivalent to its completion, but the question appears to be open in general ... it seems plausible that the answer is negative". In the present paper we establish Segal's conjecture by exhibiting an example of a non-localizable measure space whose completion is localizable.

1. Some notions from measure theory

As a general reference we use the fundamental paper [S]. For the sake of the reader's convenience and since we slightly change some of the definitions of [S] we compile here the measure theoretic notions used in section 2.

A conditional σ-ring $\mathscr{R}$ of subsets of a set R is a ring of subsets of R that is closed under countable intersection.[1] A measure space is a triple $M=(R, \mathscr{R}, r)$ which consists of a set R, a conditional σ-ring $\mathscr{R}$ of subsets of R and a finite non-negative real valued function r defined on $\mathscr{R}$ such that if $\{E_n\}$ is a sequence of mutually disjoint elements of $\mathscr{R}$ and $E=\bigcup_n E_n$ belongs to $\mathscr{R}$ then $r(E)=\sum_n r(E_n)$. A subset F of R is said to be measurable if $F\cap E\in\mathscr{R}$ for all $E\in\mathscr{R}$. The measure of a measurable set F is the least upper bound of the values of r on all those elements of $\mathscr{R}$ which are subsets of F.[2] The measure as a set function extends r and is denoted also by r. A subset F of R is called a null set if it is measurable and $r(F)=0$. The measure

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[1]) This definition is different from but equivalent to Definition 2.1 in [S].

[2]) This definition is different from Definition 2.1 in [S]. However, we are going to show that it is basicly the same. To this end define $\mathscr{R}'$ as the collection of all sets E for which there exists a sequence $\{E_n\}$ of elements of $\mathscr{R}$ such that $E=\bigcup_n E_n$ and $\sum_n r(E_n)<+\infty$. It is immediate that $\mathscr{R}'$ is a conditional σ-ring. For each element E of $\mathscr{R}'$ let $r'(E)=\sup\{r(F)\colon F\in\mathscr{R},\ F\subseteq E\}$. Then $(R, \mathscr{R}', r')$ is a measure space in the sense of Definition 2.1 of [S]. Furthermore, it is easy to verify that the measurable sets and their measures are the same in $(R, \mathscr{R}', r')$ as in $(R, \mathscr{R}, r)$.

space M is complete, by definition, if any subset of a null set in $\mathscr{R}$ is a null set in $\mathscr{R}$, or, equivalently, if any subset of a null set is a null set. The completion of M is, by definition, the measure space $M_c=(R, \mathscr{R}_c, r_c)$ where $\mathscr{R}_c$ consists of all those subsets E of R for which there exists an element F of $\mathscr{R}$ such that $(E-F)\cup(F-E)$ is contained in some element of $\mathscr{R}$ of measure zero, and then $r_c(E)=r(F)$. It can be shown by routine methods that in any measure space the collection of all measurable sets is a complemented Boolean σ-ring on which the measure is countably additive. The measure ring $\mathscr{M}$ of a measure space $M=(R, \mathscr{R}, r)$ is, by definition, the quotient of the ring of all measurable sets in M by the ideal of null sets.[3] It is clear that $\mathscr{M}$ is a complemented Boolean σ-ring. Two measure spaces are said to be strongly equivalent if their measure rings are isomorphic as Boolean rings. A measure space is called localizable if its measure ring is complete, i.e., every subset of it has a least upper bound.

2. A non-localizable space whose completion is localizable

Let I be any non-empty set and $R=I\times[0, 1]$ where $[0, 1]$ denotes the unit interval of reals. Let $\mathscr{R}$ be the collection of all subsets of R of the form[4]

$$E = [\bigcup_{i\in J} \{i\}\times E_i]\cup[\bigcup_{x\in T}\ \bigcup_{i\in K(x)} \{(i, x)\}]$$

where T is a finite subset of $[0, 1]$; the set J is a finite subset of I; for each $x\in T$ the set $K(x)$ is a co-countable subset of I (i.e. $I-K(x)$ is countable) and E_i is a Lebesgue-measurable subset of $[0, 1]$ for all $i\in J$. It is easy to verify that $\mathscr{R}$ is a conditional σ-ring. The equality $r(E)=\sum_{i\in J}$ mes (E_i) defines a countably additive finite positive measure on $\mathscr{R}$.

We are going to show that if the cardinality of I is greater than that of the continuum then the measure space $M=(R, \mathscr{R}, r)$ is non-localizable. To this end let L be a subset of I such that card L=card $(I-L)$. Denote by θ the canonical mapping of the set of all measurable sets of M onto the measure ring $\mathscr{M}$ of M. We show that $\mathscr{N}=\{\theta(\{i\}\times[0, 1]): i\in L\}$ does not have a least upper bound in $\mathscr{M}$. Suppose the contrary and denote by F an arbitrary but fixed representative of the l.u.b. of $\mathscr{N}$.

Let H be the set of all those i's ($i\in I$) for which there exists an $x\in[0,1]$ such that $(i, x)\in F$ and $(j, x)\in F$ for only countably many j's ($j\in I$). It is obvious that card H does not exceed the cardinality of the continuum. Hence on account of the assumption

[3]) Differently from Definition 2.4 in [S] we do not define any measure on $\mathscr{M}$.

[4]) We write $\{i\}$ for the singleton which contains i and (i, x) for the ordered pair whose first and second elements are i and x, respectively.

on the cardinality of L there exists an element r of L such that for every $x \in [0, 1]$ the relation $(r, x) \in F$ implies that $(i, x) \in F$ for non-countably many i's $(i \in I)$. Let X be the set of those x's, $x \in [0, 1]$, for which $(i, x) \in F$ for non-countably many i's Then $(\{r\} \times [0, 1]) \cap F \subseteqq \{r\} \times X$. We are going to show that X has Lebesgue measure zero which contradicts the fact that $\theta(F)$ is an upper bound of $\mathcal{N}$.

The measurability of F implies that X equals the set of those x's, $x \in [0, 1]$ for which $(i, x) \in F$ for co-countably many i's in I. On account of the assumption on the cardinality of $I - L$ there exists an element s of $I - L$ such that $\{s\} \times X \leqq F$. On the other hand, $r(F \cap (\{s\} \times [0, 1])) = 0$ because $s \in I - L$ and F is a representative of the *least* upper bound of $\mathcal{N}$. This implies that X has Lebesgue measure zero.

The completion of M is strongly equivalent to the direct sum $\bigoplus_{i \in I} [0, 1]$ ($[0, 1]$ with Lebesgue measure) which is localizable.[5] Since M is not localizable this implies that M cannot be strongly equivalent to its completion.

Reference

[S] I. E. SEGAL, Equivalences of measure spaces, *Amer. J. Math.*, 73 (1951), 275—313.

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[5]) Concerning the notion of direct sum and the fact that any direct sum of finite measure spaces is localizable we refer the reader to [S]. However, one can see directly that the completion under consideration is localizable.