On nonorthogonal decompositions of certain contractions

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SZ.-NAGY and FOIAŞ showed in [4] that a contraction T on a separable Hilbert space H is similar to a unitary operator if and only if its characteristic function $\Theta_T(\lambda)$ has a bounded analytic inverse (see also [5], Ch. IX). In the present paper, we give a generalization of this result. We prove that a contraction T is similar to a direct sum of a unitary operator and a contraction of class $C_{.0}$ if and only if the outer factor of $\Theta_T(\lambda)$ has a bounded analytic inverse. We shall also indicate some interesting consequences.

1. Preliminaries. We only consider non-trivial, complex, separable Hilbert spaces. For completely non-unitary contractions we will use the functional models as developed in [5], Ch. VI.

Let T be a contraction on the Hilbert space H. Denote by $D_T=(1-T^*T)^{1/2}$, $D_{T^*}=(1-TT^*)^{1/2}$ the defect operators and $\mathfrak{D}_T=\overline{D_T H}$, $\mathfrak{D}_{T^*}=\overline{D_{T^*} H}$ the defect spaces of T.

The characteristic function $\{\mathfrak{D}_T, \mathfrak{D}_{T^*}, \Theta_T(\lambda)\}$ of T is the purely contractive analytic function from $\mathfrak{D}_T$ to $\mathfrak{D}_{T^*}$ defined by

$$\Theta_T(\lambda) = [-T+\lambda D_{T^*}(1-\lambda T^*)^{-1}D_T]|\mathfrak{D}_T \quad \text{for} \quad |\lambda|<1.$$

If T is completely non-unitary, we will consider T in its functional model, i.e. defined by

$$T^*(u\oplus v) = e^{-it}[u(e^{it})-u(0)]\oplus e^{-it}v(t)$$

on the space

$$H = [H^2(\mathfrak{D}_{T^*})\oplus\overline{\Delta_T L^2(\mathfrak{D}_T)}]\ominus\{\Theta_T u\oplus\Delta_T u:\ u\in H^2(\mathfrak{D}_T)\},$$

where $\Delta_T(t)=[I-\Theta_T(e^{it})^*\Theta_T(e^{it})]^{1/2}$. Let $\Theta_T(\lambda)=\Theta_2(\lambda)\Theta_1(\lambda)$ be the canonical factorization of $\{\mathfrak{D}_T, \mathfrak{D}_{T^*}, \Theta_T(\lambda)\}$ into the product of its outer factor $\{\mathfrak{D}_T, \mathfrak{F}, \Theta_1(\lambda)\}$ and inner factor $\{\mathfrak{F}, \mathfrak{D}_{T^*}, \Theta_2(\lambda)\}$. Let

$$H_1 = \{\Theta_2 u\oplus v:\ u\in H^2(\mathfrak{F}),\ v\in\overline{\Delta_T L^2(\mathfrak{D}_T)}\}\ominus\{\Theta_T w\oplus\Delta_T w:\ w\in H^2(\mathfrak{D}_T)\}$$

be the induced invariant subspace for T and

$$H_2 = H\ominus H_1 = [H^2(\mathfrak{D}_{T^*})\ominus\Theta_2 H^2(\mathfrak{F})]\oplus\{0\}$$

its orthogonal complement. Let $T=\begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix}$ be the triangulation of T corresponding to the decomposition $H=H_1\oplus H_2$. Recall that a contraction T is of class $C_{\cdot 0}(C_{0\cdot})$ if $T^{*n}h\to 0$ $(T^n h\to 0)$ for *all* h, of class $C_{\cdot 1}(C_{1\cdot})$ if $T^{*n}h\to 0$ $(T^n h\to 0)$ for $h=0$ *only* and that $C_{\alpha\beta}=C_{\alpha\cdot}\cap C_{\cdot\beta}$ $(\alpha, \beta=0, 1)$. Note that in our case T_1, T_2 are of class $C_{\cdot 1}$, $C_{\cdot 0}$, respectively.

2. Main theorem. The main purpose of this paper is to prove the following

Theorem 1. *Let T be a completely non-unitary contraction on the separable Hilbert space $H(\neq\{0\})$ with the characteristic function $\Theta_T(\lambda)$. Let $\Theta_T(\lambda)=\Theta_2(\lambda)\Theta_1(\lambda)$ be the canonical factorization of $\Theta_T(\lambda)$ into the product of its outer factor $\Theta_1(\lambda)$ and inner factor $\Theta_2(\lambda)$. Let $T=\begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix}$ be the triangulation of T corresponding to the decomposition $H=H_1\oplus H_2$ induced by $\Theta_T(\lambda)=\Theta_2(\lambda)\Theta_1(\lambda)$. Then the following conditions are equivalent:*

(1) *T is similar to a direct sum of a unitary operator and a contraction of class $C_{\cdot 0}$;*

(2) *T_1 is similar to a unitary operator;*

(3) *$\Theta_1(\lambda)$ has a bounded analytic inverse.*

If this is the case, T is similar to $T_1\oplus T_2$.

Proof.

(1)$\Rightarrow$(2):

Assume T is similar to $U\oplus V$ on the space $K=K_1\oplus K_2$, where U is unitary on K_1 and V is a contraction of class $C_{\cdot 0}$ on K_2. Let S be an invertible operator from H onto K such that $T=S^{-1}(U\oplus V)S$. Consider $H_1'=S^{-1}K_1$ and $H_2'=H\ominus H_1'$. Obviously H_1' is an invariant subspace for T.

Let $T=\begin{bmatrix} T_1' & X' \\ 0 & T_2' \end{bmatrix}$ be the triangulation of T corresponding to the decomposition $H=H_1'\oplus H_2'$ and $S=\begin{bmatrix} S_1 & Y \\ 0 & S_2 \end{bmatrix}$ the triangulation corresponding to $H=H_1'\oplus H_2'$ and $K=K_1\oplus K_2$. Note that S_2 is invertible since S and S_1 both are and the inverse of S is given by $S^{-1}=\begin{bmatrix} S_1^{-1} & -S_1^{-1}YS_1^{-1} \\ 0 & S_2^{-1} \end{bmatrix}$. We have

$$T=\begin{bmatrix} T_1' & X' \\ 0 & T_2' \end{bmatrix}=\begin{bmatrix} S_1^{-1} & -S_1^{-1}YS_2^{-1} \\ 0 & S_2^{-1} \end{bmatrix}\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}\begin{bmatrix} S_1 & Y \\ 0 & S_2 \end{bmatrix}=\begin{bmatrix} S_1^{-1}US_1 & * \\ 0 & S_2^{-1}VS_2 \end{bmatrix}.$$

It follows that $T_1'=S_1^{-1}US_1$ and $T_2'=S_2^{-1}VS_2$. T_1' and T_2' are of class $C_{\cdot 1}$, $C_{\cdot 0}$, respectively, since U and V are. It follows from the uniqueness of the triangulation of a contraction of type $\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix}$ that H_1', H_2', T_1' and T_2' must coincide with H_1, H_2, T_1 and T_2, respectively (see [5], Sec. II. 4). Hence $T_1=S_1^{-1}US_1$ and $T_2=S_2^{-1}VS_2$. In particular, T_1 is similar to a unitary operator and T is similar to $T_1\oplus T_2$.

(2)⇔(3):

Since the characteristic function of T_1 is the purely contractive part of $\Theta_1(\lambda)$, say $\Theta_1^0(\lambda)$, and $\Theta_1(\lambda)$ has a bounded analytic inverse if and only if $\Theta_1^0(\lambda)$ has, the equivalence of (2) and (3) follows from a theorem of Sz.-Nagy and Foiaş [5, Sec. IX. 1].

(3)⇒(1):

Assume $\Theta_1(\lambda)$ has a bounded analytic inverse $\Theta_1^{-1}(\lambda)$. We will work on the functional model of T.

Let S_1 be the operator from H_1 to $\overline{\Delta_1 L^2(\mathfrak{D}_T)}$ defined by

$$S_1(\Theta_2 u\oplus v) = v-\Delta_1\Theta_1^{-1}u \quad \text{for} \quad \Theta_2 u\oplus v\in H_1$$

and S the operator from H_2 to $\overline{\Delta_1 L^2(\mathfrak{D}_T)}$ defined by

$$S(u\oplus 0) = -\Delta_1\Theta_1^{-1}\Theta_2^* u \quad \text{for} \quad u\oplus 0\in H_2$$

where $\Delta_1(t)=[I-\Theta_1(e^{it})^*\Theta_1(e^{it})]^{1/2}$. Let U be multiplication by e^{it} on the space $\overline{\Delta_1 L^2(\mathfrak{D}_T)}$. Note that U is a unitary operator.

We want to show

(i) S_1 is invertible and $S_1^{-1}US_1=T_1$, (ii) $US-ST_2=S_1X$.

For the proof of (i), consider the space

$$H_1' = [H^2(\mathfrak{F})\oplus\overline{\Delta_1 L^2(\mathfrak{D}_T)}]\ominus\{\Theta_1 w\oplus\Delta_1 w\colon w\in H^2(\mathfrak{D}_T)\}$$

and the unitary operator W from H_1 to H_1' defined by

$$W(\Theta_2 u\oplus v) = u\oplus v \quad \text{for} \quad \Theta_2 u\oplus v\in H_1$$

(see [5], p. 290, proof of Prop. VII. 2. 1). Let $S_1'=S_1W^{-1}$, i.e. S_1' is the operator from H_1' to $\overline{\Delta_1 L^2(\mathfrak{D}_T)}$ given by $S_1'(u\oplus v)=v-\Delta_1\Theta_1^{-1}u$. Now it suffices to show S_1' is invertible. Let P be the restriction to H_1' of the orthogonal projection onto $\overline{\Delta_1 L^2(\mathfrak{D}_T)}$, i.e.

$$P(u\oplus v) = v \quad \text{for} \quad u\oplus v\in H_1'.$$

By our assumption, T_1 is similar to a unitary operator. It follows that P is an invertible operator and $P^*U=(WT_1W^{-1})P^*$ (see [5], p. 342, proof of Theorem IX. 1.2). We want to show $S_1'=(P^{-1})^*$. Equivalently,

$$(P^{-1}(v'), u\oplus v) = (v', S_1'(u\oplus v))$$

for any $u\oplus v\in H_1'$ and $v'\in\overline{\Delta_1 L^2(\mathfrak{D}_T)}$, where (,) denotes the corresponding inner product. Set $P^{-1}(v')=u'\oplus v'\in H_1'$. The last equation will be the same as

$$(u'\oplus v', u\oplus v) = (v', v-\Delta_1\Theta_1^{-1}u),$$

i.e.

$$(u', u)+(v', \Delta_1\Theta_1^{-1}u) = 0.$$

But $w=\Theta_1^{-1}u\in H^2(\mathfrak{D}_T)$ so that

$$(u', u)+(v', \Delta_1\Theta_1^{-1}u) = (u', \Theta_1 w)+(v', \Delta_1 w) = 0,$$

since $u'\oplus v'\in H_1'$.

Hence we proved $S_1'=(P^{-1})^*$ is invertible and satisfies $US_1'=S_1'WT_1W^{-1}$. Since $S_1'=S_1W^{-1}$, we have S_1 is invertible and $US_1W^{-1}=US_1'=S_1'WT_1W^{-1}=$ $=S_1T_1W^{-1}$. Hence $US_1=S_1T_1$, or $S_1^{-1}US_1=T_1$ as asserted.

Now we verify (ii).

Consider any $u\oplus 0\in H_2$. Then $T(u\oplus 0)=(e^{it}u\oplus 0)-(\Theta_T w\oplus\Delta_T w)$ for some $w\in$ $\in H^2(\mathfrak{D}_T)$ and $T_2(u\oplus 0)=T(u\oplus 0)-(\Theta_2 u'\oplus v')$ for some $\Theta_2 u'\oplus v'\in H_1$. Hence

$$\begin{aligned} T_2(u\oplus 0) &= (e^{it}u\oplus 0)-(\Theta_T w\oplus\Delta_T w)-(\Theta_2 u'\oplus v')= \\ &= (e^{it}u-\Theta_T w-\Theta_2 u')\oplus(-\Delta_T w-v'). \end{aligned}$$

Since $T_2(u\oplus 0)\in H_2$, we have $-\Delta_T w-v'=0$. Note that $X(u\oplus 0)=\Theta_2 u'\oplus v'$. Hence

$$\begin{aligned} (US-ST_2)(u\oplus 0) &= U(-\Delta_1\Theta_1^{-1}\Theta_2^* u)-S[(e^{it}u-\Theta_T w-\Theta_2 u')\oplus 0] = \\ &= e^{it}(-\Delta_1\Theta_1^{-1}\Theta_2^* u)-(-\Delta_1\Theta_1^{-1}\Theta_2^*)(e^{it}u-\Theta_T w-\Theta_2 u') = \\ &= -\Delta_1\Theta_1^{-1}\Theta_2^*\Theta_T w-\Delta_1\Theta_1^{-1}u' = -\Delta_1 w-\Delta_1\Theta_1^{-1}u' = \\ &= v'-\Delta_1\Theta_1^{-1}u' = S_1(\Theta_2 u'\oplus v') = S_1X(u\oplus 0). \end{aligned}$$

This proves (ii).

Hence

$$T=\begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix}=\begin{bmatrix} S_1^{-1}US_1 & S_1^{-1}US-S_1^{-1}ST_2 \\ 0 & T_2 \end{bmatrix}=\begin{bmatrix} S_1^{-1} & -S_1^{-1}S \\ 0 & 1 \end{bmatrix}\begin{bmatrix} U & 0 \\ 0 & T_2 \end{bmatrix}\begin{bmatrix} S_1 & S \\ 0 & 1 \end{bmatrix}=$$
$$=\begin{bmatrix} S_1 & S \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} U & 0 \\ 0 & T_2 \end{bmatrix}\begin{bmatrix} S_1 & S \\ 0 & 1 \end{bmatrix}.$$

This shows T is similar to $U\oplus T_2$ on the space $\overline{\Delta_1 L^2(\mathfrak{D}_T)}\oplus H_2$ and completes the proof.

For a geometric and simpler proof of more general facts than the implication (2)⇒(1) see [2]. However the above proof gives explicit forms for the operators which implement the similarities.

3. Some consequences. An immediate result of the preceding theorem is

Theorem 2. *Let T be as in Theorem 1. Then the following conditions are equivalent:*

(1) *T is similar to an isometry;*

(2) *T_1 is similar to a unitary operator and T_2 is similar to a unilateral shift;*

(3) *$\Theta_1(\lambda)$ has a bounded analytic inverse and $\Theta_2(\lambda)$ has a bounded analytic left-inverse.*

If this is the case, T is similar to $T_1\oplus T_2$.

Proof. Since T_2 is of class $C_{\cdot 0}$, it is similar to a unilateral shift if and only if $\Theta_2(\lambda)$ has a bounded analytic left-inverse (see [6], Theorem 2.4).

This gives characterizations of those c.n.u. contractions which are similar to isometries. Another one is given by Sz.-Nagy and Foiaş [6], which says T is similar to an isometry if and only if $\Theta_T(\lambda)$ has a bounded analytic left-inverse.

In order to prove the next theorem, we need the following

Lemma. *A c.n.u. normal contraction is of class C_{00}.*

Proof. Assume T is a c.n.u. normal contraction. Then $D_T = D_{T^*}$ and

$$\Theta_T(\lambda) = [-T+\lambda D_T(1-\lambda T^*)^{-1}D_T]|\mathfrak{D}_T = (\lambda - T)(1-\lambda T^*)^{-1}|\mathfrak{D}_T,$$

which is obviously both inner and *-inner. Hence T is of class C_{00} (see [5], Sec. VI. 3).

Now we can give

Theorem 3. *Let T be as in Theorem 1. Then T is similar to a normal operator if and only if T_1 is similar to a unitary operator and T_2 is similar to a normal operator. If this is the case, T is similar to $T_1 \oplus T_2$.*

Proof. A normal contraction can be decomposed as the direct sum of a unitary operator and a c.n.u. normal contraction, the latter being of class C_{00} by the preceding lemma. The conclusion now follows from Theorem 1.

Recall that a contractive analytic function $\{\mathfrak{D}_1, \mathfrak{D}_2, \Theta(\lambda)\}$ is said to have the scalar multiple $\delta(\lambda)$, if $\delta(\lambda)$ is a scalar valued analytic function, $\delta(\lambda) \not\equiv 0$, and there exists a contractive analytic function $\{\mathfrak{D}_2, \mathfrak{D}_1, \Omega(\lambda)\}$ such that

$$\Omega(\lambda)\Theta(\lambda) = \delta(\lambda)I_{\mathfrak{D}_1}, \quad \Theta(\lambda)\Omega(\lambda) = \delta(\lambda)I_{\mathfrak{D}_2} \quad \text{for} \quad \lambda \in D.$$

A slightly different argument gives the following

Theorem 4. *Let T be as in Theorem 1. Assume, moreover, $\Theta_1(\lambda)$ admits a scalar multiple. Then T is similar to a hyponormal operator if and only if T_1 is similar to a unitary operator and T_2 is similar to a hyponormal operator. If this is the case, T is similar to $T_1 \oplus T_2$.*

Proof. We have only to prove the necessity part.

Assume T is similar to the hyponormal operator A on the space K. Let S be an invertible operator from H onto K such that $T = S^{-1}AS$. Consider $K_1 = SH_1$ and $K_2 = K \ominus K_1$. Let $A = \begin{bmatrix} A_1 & X \\ 0 & A_2 \end{bmatrix}$ be the triangulation of A corresponding to the decomposition $K = K_1 \oplus K_2$. As before, we can show T_1 is similar to A_1 and T_2 is similar to A_2. Since T_1 is of class $C_{\cdot 1}$ whose characteristic function $\Theta_1^0(\lambda)$ admits a scalar multiple (cf. [5], Prop. V. 6.8), the spectrum $\sigma(T_1)$ is contained in the unit circle C (see [5], Prop. VI. 4.4). Hence the hyomormal operator A_1 has spectrum $\sigma(A_1)=\sigma(T_1)\subseteqq C$ of planar measure zero. It follows from a theorem of PUTNAM [3], that A_1 is indeed a unitary operator. Since A is hyponormal, this will imply that K_1 is a reducing subspace for A (see, e.g., [1], Sec. 0, Ex. 2). Therefore $X=0$ and A_2 is hyponormal. This completes the proof.

Note that in the preceding theorem, if we replace "hyponormal operator" by "subnormal operator", the corresponding conclusion is still true.

The theorems are stated for c.n.u. contractions, although they are still true for arbitrary contractions; however the proof of the general case along the above lines will involve some technical difficulties.

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$$\Phi(\lambda)\Theta_2(\lambda)+\Theta_1(\lambda)\Psi(\lambda) = I.$$

Added in proof. After the paper was submitted, C. R. PUTNAM (Hyponormal contractions and strong power convergence, *Pacific J. Math.*, **57** (1975), 531—538) showed that a c.n.u hyponormal contraction is of class $C_{\cdot 0}$. Hence it follows easily that Theorem 4 here holds even without assuming $\Theta_1(\lambda)$ admits a scalar multiple.

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