On hereditarily finitely based varieties of semigroups

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Dedicated to Prof. H. Grell on his 70th birthday

In the investigation of the lattice of semigroup varieties one of the basic questions is the description of those varieties which have the property that all their subvarieties are finitely based. In particular, this means finding finite systems of identities which cannot be extended to infinite independent systems. P. PERKINS [4] has shown that commutativity has this propety and the author of the present paper has extended this result to a large class of permutative identities [5] (as L. SHEVRIN informed me recently the same generalization had been obtained by A. AĬZENŠTAT).*) However, E. LYAPIN has disappointed those who had hoped to go far by this way: in [3] he has shown that "most" balanced identities can be included in infinite independent systems. Here we continue the work in this direction: we are going to show that semigroup identities which define hereditarily finitely based varieties belong to a few exceptional types.

§ 1. The main results

Consider the free semigroup F and the free monoid F^0 over a countably infinite alphabet $X=\{x_i|i=1, 2, \dots\}$. The elements of X will be occasionally denoted also by x, y, z, y_i, z_i. A *word* is an element of F^0, its identity element being the empty word $\emptyset$. If $u, v\in F^0$ and there exist two further words u', u'' such that $v=u'uu''$ then u is a *part* of v. If $u'=\emptyset$ or $u''=\emptyset$ then u is a *beginning part* or an *end part*, respectively.

Define the quasi-order $\lhd$ on F by

Definition 1. $u\lhd v$ $(u, v\in F)$ iff there exists an endomorphism $\varphi\colon F\to F$ such that $u\varphi$ is a part of v.

If we denote the length of a word w by $l(w)$ then $u\lhd v$ obviously implies $l(u)\leqq l(v)$. — Now we extend the relation $\lhd$ to subsets of F.

*) Remark at sheet-proof: As a matter of fact, this result has been obtained already by PUTCHA and YAQUB (*Semigroup Forum*, 3 (1971), 68—73) although they have not formulated it explicitly.

Definition 1′. $U \lhd V$ ($U, V \subseteqq F$) iff $u \lhd v$ for at least one pair $(u, v) \in U \times V$. In this case we say that V depends on U; in the oppositive case it is *independent from* U.

Note that for subsets the relation $\lhd$ is not a quasi-order. — If $U = \{u\}$ we shall write also $u \lhd V$.

Definition 2. The set $V \subseteqq F$ is *dependent* (in itself) if $v \lhd V \setminus \{v\}$ for some $v \in V$, and *independent* in the opposite case.

Definition 3. A finite subset $U \subset F$ is an *essentially finite* set (EFS) if $U \lhd V$ holds for every infinite independent $V \subseteqq F$.

The significance of these notions for our purpose is established by the following propositions.

Proposition 1. *If the (finite or infinite) system of non-trivial semigroup identities*

$$(\tau) \qquad v_{2k-1} = v_{2k} \qquad (k = 1, 2, \ldots)$$

is such that $V = \{v_i | i = 1, 2, \ldots\}$ *is an independent set of words then* (τ) *is an independent system of identities.*

Proof. Denote by K_i the characteristic ideal of F generated by all v_j's, $j \neq i$, i.e. the ideal generated by $\{v_j \varphi | j \neq i, \varphi \in \mathrm{End}\,(F)\}$. It is easy to see that for every homomorphism χ: $F \to F/K_i$ we have $v_j \chi = 0$ for $j \neq i$ so that all identities of (τ) but the one containing v_i hold in F/K_i. On the other hand, the independence of V means that $v_i \notin K_i$ so that the element $\overline{v_i}$ of F/K_i corresponding to v_i under the natural homomorphism is not 0. Thus, no identity in (τ) follows from the others.

Proposition 2. *Let* (τ) *be as in Proposition* 1 *but infinite and*

$$(\sigma) \qquad u_s = u_s' \qquad (s = 1, \ldots, m)$$

arbitrary. If $V = \{v_i | i = 1, 2, \ldots\}$ *is independent from* $U = \{u_i, u_i | 1 \leqq i \leqq m\}$ *then* $(\sigma) \cup (\tau)$ *has no finite basis.*

Indeed, as above, none of the identities in (τ) follows from the rest of $(\sigma) \cup (\tau)$. However, if a finite basis existed, it could be chosen as a subsystem of $(\sigma) \cup (\tau)$.

Corollary. *If* $\mathfrak{S}$ *is a hereditarily finitely based variety of semigroups and* (σ) *is a basis of* $\mathfrak{S}$ *then* U *is an* EFS.

Indeed, in the opposite case an infinite independent system V would exist such that U non $\lhd V$ and the subvariety of $\mathfrak{S}$ defined by $(\sigma) \cup (\tau)$ would have no finite basis in virtue of Proposition 2.

By this Corollary, if we succeeded in determining all essentially finite subsets of F we could attain a considerable restriction of the scope of varieties which may be hereditarily finitely based. In this paper we determine all EFS's with 1 or 2 elements.

It holds obviously:

Proposition 3. *If* $U \subseteqq U'$ *and* U *is an* EFS *then so is* U'.

More generally,

Proposition 4. *If U is an* EFS *and U' is a finite set of words such that $U' \lhd \{u\}$ for every $u \in U$ then U' is an* EFS, *too.*

Indeed, every set depending on U depends on U'; so do all infinite independent sets.

In what follows we need some further definitions.

Definition 4. Two words u, v are *relatively prime* if no letter occurs in both of them.

Definition 5. u is a *closed part* of v if $v = u'uu''$ and u is relatively prime to u', u''. If $u \in X$ it is said to be a *closed letter.*

Definition 6. The decomposition $v = v_1 \dots v_k$ is *closed* if every pair $v_i, \ v_j$ $(i \neq j)$ is relatively prime. (Notation: $v \doteq v_1 \dots v_k$.)

Definition 7. A *type* T is a subset of F consisting of all automorphic images of any of its own elements.

In other words T consists of all elements of F which differ from each other only by the notation of letters. If $u \in T$ we shall write also $T = T(u)$.

Definition 8. A word is *simple* if all its letters are closed (i.e. if they are all distinct). Denote the set of nonempty simple words by X^*.

Definition 9. Let T be a type. The word v is *T-simple* if $v \doteq \prod_{i=1}^{n} v_i$ where $v_i \in T \cup X^*$. Denote the set of all nonempty T-simple words by T^*.

In what follows, for $U \subseteq F$ we shall denote $U \cup \{\emptyset\}$ by U^0. Put furthermore $T_0 = \left\{ v \middle| v \doteq \prod_{i=1}^{k} v_i,\ v_i \in T \right\}$ and $T_1 = T^* \setminus T_0$.

Now we formulate the main results of our paper.

Theorem 1. *The one-element set $\{u\}$ is an* EFS *iff $u \in T(xyx) \cup X^*$.*

Theorem 2. *The two-element set U is an* EFS *iff its elements are of type T and T', respectively, where one of the following cases holds (v, v', w, w' always denote closed parts of the corresponding words, $w, w' \in X^{*0}$):*

(a)	$T = T(w)$,	T' *arbitrary*,	$w \neq \emptyset$;
(b)	$T = T(xyx)$,	T' *arbitrary*;	
(c)	$T = T(xwx)$,	$T' = T(x_1 y_1 x_1 w' x_2 y_2 x_2)$,	$w, w' \neq \emptyset$;
(d)	$T = T(xyzx)$,	$T' = T(vwv')$,	$w \neq \emptyset,\ v, v' \in T(x^2)^0$;
(d′)	$T = T(xyzx)$,	$T' = T(x^2)$;	
(e_1)	$T = T(xyzx)$,	$T' = T(x^2 wyzy)$;	
(e_2)	$T = T(xyzx)$,	$T' = T(yzywx^2)$;	
(f)	$T = T(wxyxw')$,	$T' = T(v)$,	$v \in T^*(x^2)$;

(g_1)	$T=T(xyxw)$,	$T'=T(xvyx)$,	$v\in T^*(x^2)$;
(g_2)	$T=T(wxyx)$,	$T'=T(xyvx)$,	$v\in T^*(x^2)$;
(h_1)	$T=T(xyxz)$,	$T'=T(x\bar{v}x)$,	$v\in T_1(x^2)$;
(h_2)	$T=T(zxyx)$,	$T'=T(xvx)$,	$v\in T_1(x^2)$.

§ 2. Only if

In order to prove the "only if" parts we are going to list eight infinite independent word sets (Proposition 5); we shall see that all one- and two-element sets which the infinite subsets of the sets $V_1, \ldots, V_6''$ depend on are those mentioned in the theorems. — For not to be obliged to prove the independence of each V_i separately we shall use the following

Lemma. *Let u, u' and w be words having the following properties:*

i) *their first letters coincide with the last ones;*

ii) $l(w)>1$;

iii) *if* $l(u)>1$ $(l(u')>1)$ *then* u non $\lhd$ w (u' non $\lhd$ w);

iv) *if* $l(u)=1$ $(l(u')=1)$ *then the letter* $u(u')$ *occurs in* $u'(u)$.

Then the set

$$V=\left\{v_n|v_n = u\left(\prod_{i=1}^{n} w_i\right)u',\ w_i\in T(w),\ w_i\ \textit{closed}\right\}$$

is independent.

Sketch of the proof. Suppose $v_n\varphi$ is a part of v_m for $n<m$. First one shows that $u\varphi=u$ and $u'\varphi=u'$ (this follows from i, iii, iv and the fact that w_i is closed); hence $v_n\varphi=v_m$ and $l(w_k\varphi)>l(w)$ for some $k\leqq n$. However, this is impossible by i, ii and the same fact as before.

Now we obtain immediately

Proposition 5. *Put* $a_n=\prod_{i=1}^{n} x_{2i-1}x_{2i}x_{2i-1}$, $b_n=\prod_{i=1}^{n} x_i^2$. *Then the sets*

$V_1 = \{x_1 \ldots x_{n-1}\, x_n x_{n-1} \ldots x_1 x_n \mid n=3, 4, \ldots\}$,

$V_2 = \{y_1 z_1 z_2 y_1 a_n y_2 z_3 z_4 y_2 \mid n=1, 2, \ldots\}$,

$V_3 = \{y a_n y \mid n=1, 2, \ldots\}$,

$V_4 = \{y_1^2 a_n y_2^2 \mid n=1, 2, \ldots\}$,

$V_5 = \{y_1 z_1 y_1 b_n y_2 z_2 y_2 \mid n=1, 2, \ldots\}$,

$V_6 = \{y b_n y \mid n=1, 2, \ldots\}$,

$V_6' = \{yzyb_n z \mid n=1, 2, \ldots\}$,

$V_6'' = \{z b_n yzy \mid n=1, 2, \ldots\}$

are independent.

Proof. For V_1 this is known [1], for the rest it follows from the Lemma.

Before reverting to our Theorems, we state some simple propositions which we shall use later without refering to them.

Proposition 6. *If* $u \in X^*$ *then* $u \lhd v$ *iff* $l(u) \leqq l(v)$.

Proposition 7. xyx non $\lhd$ v *iff* $v \in T^*(x^2)$.

Proposition 8. $v \in T^*(xyx)$ *iff* x^2 non $\lhd$ v, $xyzx$ non $\lhd$ v.

Propisition 9. *If* $u\varphi = v_1 v_2 v_3$, v_2 *closed, then* $u = u_1 u_2 u_3$, u_2 *closed,* $u_i \varphi = v_i$ $(i = 1, 2, 3)$.

Now we can prove the "only if" parts of both theorems.

Necessity (Theorem 1). If $\{u\}$ is essentially finite and u is not simple then $u \lhd V_1$ implies x^2 non $\lhd$ u. Hence, by $u \lhd V_6$, u must be of the form xwx, w simple. However $u \lhd V_4$ implies $l(w) = 1$.

Necessity. (Theorem 2). Suppose $U = \{u_1, u_2\}$ is a minimal EFS (i.e. $u_1, u_2 \notin T(xyx) \cup X^*$). Put first $x^2 \lhd u_2$. Then $U \lhd V_1$ implies x^2 non $\lhd$ u_1. By the same reason, no letter can occur in u_1 more than twice. Moreover, $u_1 \lhd V_1$, $u_1 \lhd V_2$ imply that at most (and then, by minimality of U, exactly) one letter may occur twice. Indeed, if x_i and x_j both occur twice in u_1 and say, the first occurance of x_i precedes that of x_j then either this latter precedes the second occurance of x_i ($u_1 = \dots x_i \dots x_j \dots \dots x_i \dots$) and u_1 non $\lhd$ V_2 or else (i.e. if $u_1 = \dots x_i \dots x_i \dots x_j \dots x_j \dots$) u_1 non $\lhd$ V_1. Thus, $u_1 = w x_i w' x_i w''$ (w, w', w'' simple and closed). Furthermore $u_1 \lhd V_2$ entails $l(w') \leqq 2$ and, since $u_1 \lhd V_3$, we have $w = w'' = \emptyset$ if $l(w') > 1$. Hence $u_1 = x_i x_j x_k x_i$ or $u_1 \doteq w x_i x_j x_i w''$.

In the first case u_1 non $\lhd$ V_4, u_1 non $\lhd$ V_5 and so $u_2 \lhd V_4$ which implies $u_2 \in T(x^2)$ — case (d') — or $u_2 \doteq v_1 q v_2$ with $v_1, v_2 \in T(x^2) \cup T(xyx) \cup X$, x^2 non $\lhd$ q, $q \neq \emptyset$ if $v_1, v_2 \in T(x^2)$, and $u_2 \lhd V_5$ which implies xyx non $\lhd$ q, $q \neq \emptyset$ if $v_1, v_2 \in T(xyx)$. Now x^2 non $\lhd$ q, xyx non $\lhd$ q give $q \in X^{*0}$ and we get either one of the cases (d), (d'), (e_1), (e_2) or a subcase of (g_1) or (g_2).

If $u_1 \doteq w x_i x_j x_i w'$ then $w = w' = \emptyset$ is impossible in virtue of the minimality of U. Suppose $w \neq \emptyset$. Then u_1 non $\lhd$ V_6, $u_2 \lhd V_6$ and either $u_2 \in T^*(x^2)$ and we obtain case (f) or $u_2 = x_{m+1} v x_{m+1}$, $v \in T^*(x^2)$, v closed. In the latter case u_2 non $\lhd$ V_5, $u_1 \lhd V_5$; hence $w' = \emptyset$. Furthermore, if $v \in T_0(x^2)$, $l(v) = 2m$ then u_2 non $\lhd$ $V_5^{(m)} = \{y b_n y \mid n > m\}$. Thus, $v \in T_1(x^2)$. Now if $l(w) = 1$ we have case (h_2); if $l(w) > 1$ then u_1 non $\lhd$ V_6, $u_2 \lhd V_6$ which implies $v \doteq y v'$ and we obtain (g_2). The case where $w' \neq \emptyset$ can be settled analogously.

Now let x^2 non $\lhd u_1$, x^2 non $\lhd u_2$. Put $u_1 \lhd V_6$; then $u_1 = x_{m+1} w x_{m+1}$, $w \in X^*$, closed, $l(w) > 1$ by the minimality of U. Now u_1 non $\lhd V_5$, and thus $u_2 \doteq wu'w'$ where $w, w' \in T(xyx)^0$, $u' \in X^*$. If $w, w' \neq \emptyset$ we have case (c), if $w = \emptyset$ or $w' = \emptyset$ we get a subcase of (g_1) or (g_2), respectively. This completes the proof of the necessity.

§ 3. If

In proving that the sets given in Theorems 1 and 2 are essentially finite we have to show that infinite sets depending on them are dependent in themselves. For this, we need some theorems which assure the dependence of certain types of infinite sets, and these theorems in their turn are based on some results in the theory of q.o. sets. We are going now to quote these latter ones.

Definition 10. The quasi-ordered set P is a *well quasiordered* set (WQOS) if it satisfies the descending chain condition and does not contain infinite independent subsets (i.e. infinite sets of pairwise incomparable elements).

Next we give some plain facts.

(I) *Let P be a quasi-ordered set. If there exists a mapping γ of P in a* WQOS *R such that $p\gamma \leqq p'\gamma$ implies $p \leqq p'$ then P is a* WQOS *itself.*

Let us mention two important particular cases:

(I_1) *A subset of a* WQOS *is a* WQOS.

(I_2) *If $\prec$ is a refinement of the q.o. $<$ on P and P is a* WQOS *under $<$ then so it is under $\prec$.*

(II) *The union of a finite number of* WQOS's *is a* WQOS.

Now let P be a q.o. set. Define a q.o. on the set $\bar{P}$ of all finite sequences of elements of P by

$$\pi = (p_1, \ldots, p_n) \leqq (p_1', \ldots, p_m') = \pi'$$

iff there exists a subsequence $(p'_{i_1}, \ldots, p'_{i_n})$, $1 \leqq i_1 < \ldots < i_n \leqq m$, of π' such that $p_j \leqq p'_{i_j}$. The following proposition is a consequence of a theorem of G. Higman [2].

(III) *If P is a* WQOS *then so is $\bar{P}$.*

We prefer to give here a self-contained proof. First of all, the classical theorem of Ramsey implies:

(IV) *The direct product of a finite number of* WQOS's *is a* WQOS.

It is routine to check the validity of the descending chain condition in $\bar{P}$. Now suppose P_1 is an infinite independent subset of $\bar{P}$ and let $\pi = (p_1, \ldots, p_n)$ be an element of minimal length in P_1. Suppose $\pi' = (p_1, \ldots, p_{k-1})$ is the maximal segment of π such that the subset $R = \{\varrho \in P_1 \mid \pi' < \varrho\}$ of P_1 is infinite; obviously, $0 < k \leqq n$. Choose a subsequence $\varrho'=(r_{\varrho(1)}, \dots, r_{\varrho(k-1)})$ in each $\varrho\in R$ such that $p_i\leqq r_{\varrho(i)}$, each $\varrho(i)$ as small as possible. By (IV) and the theorem of RAMSEY there exists an infinite subset R' of R such that the set $R''=\{\varrho' \mid \varrho\in R'\}$ is totally quasi ordered (i.e. any two elements of R'' are comparable).

Now we "break up" the elements of R': for each $\varrho\in R'$ put $\varrho^{(i)}=\{r_{\varrho(i-1)+1}, \dots, r_{\varrho(i)-1}\}$ where $\varrho(0)=0$, $\varrho(k)=l(\varrho)+1$ and $\varrho(i)$ is defined as above if $1\leqq i\leqq k-1$. It can be seen easily that every component of $\varrho^{(i)}$ $(i=1, \dots, k)$ is either strictly less then p_i or incomparable with it. At least one of the sets $R^{(i)}=\{\varrho^{(i)} \mid \varrho\in R'\}$ must contain an infinite independent subset; indeed, in the opposite case all they are WQOS's and, by (IV), so is the direct product $Q=R''\oplus R^{(1)}\oplus\dots\oplus R^{(k)}$. However then the mapping $\gamma\colon R'\to Q$ $(\varrho\gamma=(\varrho', \varrho^{(1)}, \dots, \varrho^{(k)}))$ satisfies the conditions in (I) and therefore R' is a WQOS which is a contradiction.

Suppose $R^{(i_1)}$ contains an infinite independent subset P_2. Repeating the above construction, we find a component $p_{i_1 i_2}$ of a vector of minimal length in P_2 such that there exists an infinite independent set P_3 consisting of vectors with components stricly less than or incomparable with $p_{i_1 i_2}$. Thus we obtain an infinite series p_{i_1}, $p_{i_1 i_2}$, ... of elements of P having the property that every member of it is either strictly less than or incomparable with each of the preceding ones which is impossible.

In applying these facts to word sets we shall use besides $\lhd$ three further relations:

Definition 1_l. $u\lhd_l v$ iff there exists an endomorphism $\varphi\colon F\to F$ such that $u\varphi$ is an end part of v $(v=v'\cdot u\varphi)$.

Definition 1_r. $u\lhd_r v$ iff there exists an endomorphism $\varphi\colon F\to F$ such that $u\varphi$ is a beginning part of v $(v=u\varphi\cdot v')$.

Definition 1_q. $u\lhd_q v$ iff there exists an endomorphism $\varphi\colon F\to F$ such that $v=u\varphi$.

For a set of words V the q.o. set $\{V, \lhd\}$ will be simply denoted by V; furthermore, we shall write

$$V_l=\{V, \lhd_l\}, \quad V_r=\{V, \lhd_r\}, \quad V_q=\{V, \lhd_q\}.$$

Obviously, all these sets satisfy d.c.c. By (I_2), if V_q is a WQOS then so is V_l and V_r and if either of these latter ones is a WQOS then so is V.

We introduce an operation $\circ$ on sets of words as follows:

$$U\circ V=\{w \mid w\doteq uv,\ u\in U,\ v\in V\}.$$

It holds

Theorem 3. *Let* $U, U', V_r, V_l', W_q, W_q'$ *be* WQOS's. *Then* $V'\circ V$, $(V'\circ W)_l$, $(W'\circ V)_r$ *and* $(W'\circ W)_q$ *are* WQOS's. *Moreover, if either the last letter of every element of* U *and* V *or the first letter of every element of* U' *and* V' *is closed then* $U\circ U'$, $(U\circ W')_l$. $(W\circ U')$, *and* $(V\circ V')_q$ *are* WQOS's, *too.*

Proof. Let us prove the assertion concerning $U\circ U'$; in the other cases the proof runs analogously. Suppose for example that $U=\{u_i|u_i\doteq u_i^*\ y_i,\ i=1,2,\dots\}$, $U'=\{u_j'|j=1,2,\dots\}$. Then $U\circ U'=\{w_{ij}|w_{ij}\doteq u_iu_j'\}$. Now the direct product $U\times U'$ is a WQOS by (IV). Consider the mapping $\gamma: U\circ U'\to U\times U'$ defined by $w_{ij}\ \gamma=(u_i,u_j')$. Now let $(u_i,u_j')\leqq(u_k,u_p')$ i.e. $u_i\lhd u_k$, $u_j'\lhd u_l'$ and, say, $u_k=s(u_i\varphi)t=s(u_i^*\varphi)(y_i\varphi)t$, $u_l'=s'(u_j'\psi)t'$. As u_i^* and u_j' are relatively prime and none of them contains y_i, there exists an endomorphism $\chi: F\to F$ such that $u_i^*\chi=u_i^*\varphi$, $u_j'\chi=u_j'\psi$ and $y_i\chi=$ $=y_i\varphi\cdot ts'$. Thus $w_{kl}=u_ku_l'=sw_{ij}\chi t'$, i.e. $w_{ij}\lhd w_{kl}$. Hence the assertion follows by (I).

Theorem 4. *Let T be a type. Then T^*, T_l^*, T_r^*, T_{1q} are* WQOS's.

Proof. First remark that X_q^* and T_{0r} are WQOS's. Hence, by Theorem 3 (case $(V\circ V')_q$) and (II), the posets $T_q'=(T_0\circ X^*)_q$ and $T_q''=(T'\cup X^*)_q$ are WQOS's. Using the notation $T'^*=\{w|w=\prod_{i=1}^n t_i,\ t_i\in T'\}$, we have $T_1=X^*\cup T'^*\cup(X^*\circ T'^*)\cup$ $\cup(T'^*\cup(X^*\circ T'^*))\circ T_0$. Thus, it suffices to show that $T_l'^*$ is a WQOS since then by Theorem 3 $(T'\circ T'^*)_q$ and hence $(T'\cup(T'\circ T'^*))_q=T_q'^*$ are also WQOS's. Using again Theorem 3, (I_2) and (II) we conclude, furthermore, that T_{1q} and $T_r^*=(T_0\cup T_1)_r$ are WQOS's, too. The rest follows by duality and by (I_2), respectively.

Since T_q' is a WQOS, the same follows from (III) for the set of finite sequences $\overline{T_q'}$. Consider the mapping $\gamma: T_l'^*\to\overline{T_q'}$ defined by $w\gamma\doteq\left(\prod_{i=1}^n t_i\right)\gamma=(t_1,\dots,t_n)$ $(t_i\in T')$. We have to show that γ satisfies the condition of (I). Put $w'=\prod_{j=1}^m t_j'\in T'^*$ and suppose $(t_1,\dots,t_n)\leqq(t_1',\dots,t_m')$ in $\overline{T_q'}$, i.e. $t_i\lhd_q t_{j_i}'$ for some $1\leqq j_1<\dots<j_n\leqq m$. In other words, there exist endomorphisms $\varphi_i: F\to F$ $(i=1,\dots,n)$ such that $t_i\varphi_i=t_{j_i}'$. Denote the last letter of t_i by y_i; then $t_i\doteq t_i^*y_i$. Since $t_1^*,\dots,t_n^*$ are pairwise relatively prime and they do not contain $y_1,\dots,y_n$, there exists an endomorphism $\varphi: F\to F$ such that $t_i^*\varphi=t_i^*\varphi_i$ for $i=1,\dots,n$; $y_i\varphi=y_i\varphi_i\cdot\prod_{k=j_i+1}^{j_{i+1}-1}t_k'$ for $i=1,\dots,n-1$ and $y_n\varphi=y_n\varphi_n\prod_{k=j_n+1}^{m}t_k'$. Hence $t_i\varphi=\prod_{k=j_i}^{j_{i+1}-1}t_k'$ for $i<n$, $t_n\varphi=\prod_{k=j_n}^{m}t_k'$ and $w'=\left(\prod_{k=1}^{j_1-1}t_k'\right)\times$ $\times(w\varphi)$. The theorem is proved.

Theorem 5. *Let $V=\{v_i\mid i=1,2,\dots\}$ be a set of words. Suppose there exist natural numbers k, l, n and n types $T^{(1)},\dots,T^{(n)}$ such that every v_i has a decomposition $v_i=v_{i0}\cdot\prod_{j=1}^n u_{ij}v_{ij}$ where*

a) $l(u_{ij})\leqq l$,

b) $v_{ij}\in T^{(j)*0}$,

c) *v_{ij} is closed if non-empty,*

d) *if $l(v_{ij})>k$ for some $j\neq 0,n$ then $v_{ij}\in T_1^{(j)}$.*

Then V is dependent. Moreover, if $v_{i0} \neq \emptyset$ ($v_{in} \neq \emptyset$) for every i then V_r (V_l) is dependent as well.

Proof. We shall prove that V contains an infinite WQOS. Indeed, there is only a finite number of sequences $u_1, \ldots, u_n$ of words of bounded length. Consequently, V has an infinite subset V' such that to every $v_i \in V'$ the same sequence $u_1, \ldots, u_n$ corresponds. Now $V' = V_0^* \cup V_0^{**}$ where $V_0^* = \{v_i \mid v_i \in V', v_{i0} \neq \emptyset\}$, $V_0^{**} = V' \setminus V_0^*$. Put $V_0' = V_0^*$ if V_0^* is infinite and $V_0' = V_0^{**}$ in the opposite case. Construct in a similar way consecutively $V_0' \supseteqq V_1' \supseteqq \ldots \supseteqq V_n'$ with either $V_r' = V_r^* = \{v_i \mid v_i \in V_{r-1}', v_{ir} \neq \emptyset\}$ or $V_r' = V_{r-1}' \setminus V_r^*$. For sake of simplicity suppose $u_j \neq \emptyset$ and, for $j \neq 0$, $j \neq n$, $V_j' = V_j^*$ (in the opposite case we possibly had to change the parameters k, l, n and the decomposition of v_i). Put $T_0^{(j)}(k) = \{w \mid w \in T_0^{(j)}, l(w) \leqq k\}$, $U_j = T_1^{(j)} \cup T_0^{(j)}(k)$ for $1 \leqq j \leqq n-1$, $U_0 = T^{(0)*}$ or $\{\emptyset\}$ according to $V_0' = V_0^*$ or V_0^{**} and $U_n = T^{(n)*}$ or $\{\emptyset\}$ according to $V_n' = V_n^*$ or V_n^{**}. Then $v_{ij} \in U_j$ for $v_i \in V_n'$, and U_{jq} ($1 \leqq j \leqq n-1$) as well as U_{0l}, U_{nr} (if different from $\{\emptyset\}$) are WQOS's. This implies that $A = U_{0l} \times \times \left(\prod_{j=1}^{n-1} U_{jq}\right) \times U_{nr}$ is also a WQOS. Define $\gamma\colon V_n' \to A$ by $v_i \gamma = \left(v_{i0} \cdot \prod_{j=1}^{n} u_j v_{ij}\right) \gamma = (v_{i0}, \ldots, v_{in})$. Suppose $v_h \in V_n'$ and $(v_{i0}, \ldots, v_{in}) \leqq (v_{h0}, \ldots, v_{hn})$ in A, i.e. $v_{h0} = w \cdot (v_{i0} \varphi_0)$, $v_{hn} = (v_{in} \varphi_n) \cdot w'$ (if non-empty) and $v_{hj} = v_{ij} \varphi_j$ ($j = 1, \ldots, n-1$) with some suitable endomorphisms $\varphi_0, \ldots, \varphi_n$ of F. In consequence of c), there exists $\varphi\colon F \to F$ such that $u_j \varphi = u_j$, $v_{ij} \varphi = v_{ij} \varphi_j$ so that $v_h = w(v_i \varphi) w'$. Consequently, V_n is an infinite WQOS and therefore contains comparable elements which completes the proof.

Now we are in position to prove the second parts of Theorems 1 and 2.

Sufficiency (Theorem 1). In consequence of Proposition 6, every infinite set depends on $u \in X^*$. If $u \in T(xyx)$ and u non $\lhd$ V then, by Proposition 7, $V \subseteqq T^*(x^2)$ and it is dependent in itself by Theorem 4. This completes the proof of Theorem 1.

Sufficiency (Theorem 2). We shall show that U is an EFS in cases (c), (e_1), (f), (g_1), (h_1) and in the subcase of (d) where $v, v' \in T(x^2)$. Cases (e_2), (g_2), (h_2) follow then by duality, (d') and the rest of (d) from Proposition 4, and (a), (b) from Proposition 3. In all cases the proof consists in finding the general form of words which are independent from U and in a subsequent application of Theorem 5 to infinite sets consisting of such words. Thus, put U non $\lhd$ q. If $q = q_1 x q_2$ we shall say that x occurs in q later (earlier) again if $q_2 = q' x q''$, $q' \neq \emptyset$ ($q_1 = q' x q''$, $q'' \neq \emptyset$).

Case (c). Let $q \doteq q' t q''$ where $q', q'' \in T^*(x^2)^0$ and either $t = \emptyset$ or the first letter of t occurs later again, the last one earlier again in q. We have $l(t) < 2l(w) + l(w') + 2$. Indeed, the number of letters between the first and last occurrance of the extreme letters of t cannot exceed $l(w) - 1$, and the number of those between the last occurrance of the first letter of t and the first occurance of the last one must be less than $l(w')$. Put k arbitrary, $l = 2l(w) + l(w') + 1$, $n = 1$, $T^{(0)} = T^{(1)} = T(x^2)$ and apply Theorem 5.

Case (d), $v, v' \in T(x^2)$. Let $q \doteq q'tq''$ where $q'q'' \in T^*(xyx)^0$, $t = x_i^2 t' x_j^2$ or $t \in T(x^2)^0$ (as $xyzx$ non $\lhd q$, such a decomposition exists by Proposition 8). Then $l(t') < l(w)$ and we can apply Theorem 5 with k arbitrary, $l = l(w)+3$, $n=1$, $T^{(0)} = T^{(1)} = T(xyx)$.

Case (e_1). Since $xyzx$ non $\lhd q$, there exists a decomposition $q \doteq q'tq''$ where $q' \in T^*(xyx)^0$, $q'' \in T^*(x^2)^0$ and $t = x_i^2 t' x x_j x_k x_j$ (or $t = \emptyset$). Then $l(t') < l(w)$. Put k arbitrary, $l = l(w)+4$, $n=1$, $T^{(0)} = T(xyx)$, $T^{(1)} = T(x^2)$ and Theorem 5 yields the dependence.

Case (f). If $q = q_1 x_j q_2 x_j q_3$ with $q_2 \neq \emptyset$ then either $l(q_1) < l(w)$ or $l(q_3) < l(w')$. Hence $q = tq't'$ where $l(t) \leqq l(w)$, $l(t') \leqq l(w')$ and $q' \in T^*(x^2)^0$; moreover, if $x_{i_1}, \ldots, x_{i_s}$ are the letters which occur in q' and in either t or t', too, then $q = tv_0 \left(\prod_{j=1}^{s} x_{i_j}^{\varepsilon_j} v_j \right) t'$ where $\varepsilon_j = 1$ or 2, $v_j \in T^*(x^2)^0$, v_j closed if non-empty and if $l(v_j) \geqq 2m(v)$ ($m(v)$ denotes the number of different letters in v) then $v_j \in T_1(x^2)$ (in the opposite case we had $v \lhd q$). Thus, we can apply Theorem 5 putting $k = 2m(v)$, $l = 2(l(w)+l(w'))$, some $n \leqq l(w)+l(w')+2$ and $T^{(0)} = \ldots = T^{(n)} = T(x^2)$.

Case (g_1). Here $q = v_0 \left(\prod_{j=1}^{s} x_{i_j}^{\varepsilon_j} v_j \right) t$ where $l(t) = l(w)$, $x_{i_1}, \ldots, x_{i_s}$ are different letters occurring also in t, $\varepsilon_j = 1$ or 2, $v_j \in T^*(x^2)^0$, v_j closed if non-empty and $v_j \in T_1(x^2)$ if $l(v_j) > 2m(v)$, $j \neq 0$ (in the opposite case $xvyx \lhd q$). One can apply Theorem 5 with $k = 2m(v)+1$, $l = 2l(w)$, some $n \leqq l(w)+1$ and $T^{(0)} = \ldots = T^{(n)} = T(x^2)$.

Case (h_1). Now either $q \in T^*(x^2)$ or $q = v_0 x_i v_1 x_i$ where $\varepsilon = 1$ or 2, $v_0, v_1 \in T^*(x^2)^0$, closed, and $v_1 \in T_1(x^2)$ if $l(v_1) > 2m(v)$. Again, put $k = 2m(v)$, $l = 2$, $n = 2$, $T^{(0)} = T^{(1)} = T^{(2)} = T(x^2)$ (as a matter of fact, $T^{(2)}$ is unrelevant since $U_2 = \emptyset$).

Theorem 2 is proved.

Corollary. *If the identity $u = u'$ defines a hereditarily finitely based variety then the pair of types $T(u)$, $T(u')$ is one of the pairs (a)—(h_2).*

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