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A convolution theorem and a remark on uniformly closed Fourier algebras

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Let G be a locally compact group. Recently J. T. BURNHAM and R. R. GOLDBERG [3] gave a new and elementary proof to the following theorem of DIEUDONNÉ [4]: If G is abelian and not discrete then $f * L_p(G)$ is a proper subset of $L_p(G)$ for all f in $L_1(G)$ and all $p \geqq 1$. Their proof, while simpler than Dieudonné's, relies on the structure of $L_1(G)$ as a commutative Banach algebra and therefore does not seem to extend to nonabelian groups. In the first part of this paper we prove this result for nonabelian groups. Our proof depends only on the structure of $L_p(G)$ as a Banach space and as a Banach $L_1(G)$ module. A corollary of this result is that $L_1(G)$ is not countably generated, algebraically, as a right ideal.

In part two we use a characterization of multipliers on $L_\infty(G)$ to give a new proof to the following result due to M. RAJAGAPOLAN [10] and to L. T. GARDNER [7]: If $L_1(G)$ is equivalent to a C^*-algebra then G is finite.

I. The Convolution Theorem

Let left Haar measure be denoted by μ and let Δ denote the modular function of G. If f is a function on G we denote by $\tilde{f}$ the function defined by $\tilde{f}(x)=f(x^{-1})$ for all x in G. Let $C_{00}(G)$ denote the set of continuous functions on G with compact support and $C_0(G)$ denote the set of continuous functions on G that vanish at infinity.

Lemma 1.1. *Let h belong to $C_{00}(G)$ and g belong to $L_p(G)$, $p \geqq 1$. Then $h * g$ is an everywhere defined continuous function on G.*

Proof. Let $q=p(p-1)^{-1}$ if $p \neq 1$ and let $q=\infty$ if $p=1$. Then for x in G we have

$$|h*g| = \left|\int h(y)g(y^{-1}x)\,d\mu(y)\right| = \left|\int \Delta(y^{-1})h(y^{-1})g(yx)\,d\mu(y)\right| =$$
$$= \left|\int (\Delta h)^{\sim}(y)g(yx)\,d\mu(yx)\right| \leqq \Delta(x^{-1})\|(\Delta h)^{\sim}\|_q\|g\|_p.$$

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This shows that $h*g$ is everywhere defined. A similar computation shows that for x_1 and x_2 in G we have

$$|h*g(x_1)-h*g(x_2)| \leqq |\Delta(x_1^{-1})-\Delta(x_2^{-1})| \, \|(\Delta h)^{\sim}\|_q \|g\|_p =$$

$$= |\Delta(x_1)-\Delta(x_2)| \Delta(x_1x_2)^{-1} \|(\Delta h)^{\sim}\|_q \|g\|_p .$$

The continuity of $h*g$ now follows from the continuity of Δ.

Theorem 1.2. *Let G be a locally compact, non-discrete group. Then $f*L_p(G)$ is a proper subset of $L_p(G)$ for all f in $L_1(G)$ and all $p\geqq 1$.*

Proof. Suppose $f*L_p(G)=L_p(G)$ for some f in $L_1(G)$ and some $p\geqq 1$. Then the map T_f: $L_p(G)\to L_p(G)$ defined by $T_f(g)=f*g$ is continuous and surjective. By the open mapping theorem [8, E. 2 (iii)] there exists a constant $M>0$ such that given any g in $L_p(G)$ with $\|g\|_p\leqq 1$ there exists an h in $L_p(G)$ with $\|h\|_p\leqq M$ such that $T_f(h)=g$. Choose f_0 in $C_{00}(G)$ such that $\|f-f_0\|_1\leqq(2M)^{-1}$. Consider the map T_{f_0}: $L_p(G)\to L_p(G)$ defined as above. Given any g in $L_p(G)$ with $\|g\|_p\leqq 1$, we can choose h in $L_p(G)$ with $\|h\|_p\leqq M$ and $T_f(h)=g$. Then

$$\|T_{f_0}(h)-g\|_p = \|T_{f_0}(h)-T_f(h)\|_p = \|f_0*h-f*h\|_p \leqq \|f_0-f\|_1\|h\|_p < 2^{-1}.$$

By the theorem of W. Bade and P. C. Curtis [2, Thm. 1.2] this implies that T_{f_0} maps $L_p(G)$ onto $L_p(G)$. Therefore $f_0*L_p(G)=L_p(G)$. It follows from Lemma 1.1 that every function in $L_p(G)$ is equal to a continuous function locally almost everywhere. In particular if h belongs to $L_p(G)$ and K is a compact subset of G then there exists a constant $M\geqq 0$ such that the set $\{x\in K: |h(x)|\geqq M\}$ has measure zero. Since G is not discrete we can choose a decreasing sequence $\{U_n\}_{n=1}^{\infty}$ of compact neighborhoods of the identity such that $\mu(U_n)<n^{-p-1}$ for $n=1, 2, \ldots$. Let $h(x)=\sum_{n=1}^{\infty} n\xi_{U_n}(x)$ where $\xi_{U_n}(x)$ is the characteristic function of the set U_n. Then h belongs to $L_p(G)$. But the sets $\{x\in U_1: |h(x)|\geqq n\}$ have positive measure for all $n=1, 2, \ldots$. This contradiction proves the theorem.

Corollary 1.3. *Let G be a locally compact non-discrete group and M a countable subset of $L_1(G)$. Then* $\operatorname{span}(M*L_p(G))$ *is a proper subset of $L_p(G)$ for all $p\geqq 1$.*

Proof. By [8, 32.50 (c)] there exists an f in $L_1(G)$ such that M is contained in $f*L_1(G)$. Then we have that $\operatorname{span}(M*L_p(G))\subset f*L_1(G)*L_p(G)\subset f*L_p(G)$. The corollary now follows from Theorem 1.2.

Corollary 1.3 shows, in particular, that $L_1(G)$ is not countably generated, algebraically, as a right ideal.

II. Fourier Algebras

Let $B(G)$ denote the Fourier—Steltjes algebra of G, $A(G)$ the Fourier algebra of G, $C^*(G)$ the C^*-enveloping algebra of G and $W^*(G)$ the W^*-enveloping algebra of G. For the definitions of these objects the reader is referred to [6]. It is shown in [6] that $B(G)$ is the predual of $W^*(G)$. Now $A(G)$ is a closed, translation invariant subspace of $B(G)$ and so, as noted in [12, p. 23], there exists a central projection z in $W^*(G)$ such that $A(G)$ can be identified with those f in $B(G)$ such that $f(za)=f(a)$ for all a in $W^*(G)$. We write ${}_zf$ for the functional $a\to f(za)$ on $W^*(G)$ and $A(G)={}_zB(G)$.

A multiplier on $L_\infty(G)$ is a linear operator on $L_\infty(G)$ that commutes with left translation by elements G.

Lemma 2. 1. *Suppose $L_1(G)$ is equivalent to a C^*-algebra. Then $L_\infty(G)=B(G)$ and there exists a norm continuous multiplier P from $L_\infty(G)$ onto $C_0(G)$ such that $P^2=P$.*

Proof. Let $M(G)$ denote the set of finite, regular, Borel measures on G and ω the natural embedding of $M(G)$ into $W^*(G)$, see [6 and 12]. Since ω is a *-isomorphism and $L_1(G)$ is equivalent to a C^*-algebra we have by [11, Cor. 4.8.6] that $\omega|L_1(G)$ is bicontinuous. But $\omega(L_1(G))$ is dense in $C^*(G)$ and so it follows that $\omega(L_1(G))=C^*(G)$. By taking the adjoint of $\omega|L_1(G)$ we get $B(G)$, as the dual space of $C^*(G)$, bicontinuously isomorphic to $L_\infty(G)$, as the dual space of $L_1(G)$. But then by [6, p. 193] the image of $B(G)$ under the adjoint of $\omega|L_1(G)$ is $B(G)$ as a subspace of $L_\infty(G)$. Therefore $L_\infty(G)=B(G)$.

The subspace $A(G)$ is closed in $B(G)$ and dense in $C_0(G)$ and so $A(G)=C_0(G)$. We had $A(G)={}_zB(G)$ where z was a central projection in $W^*(G)$. Let x be in G and f in $B(G)$, then ${}_z({}_{\omega(x)}f)={}_{\omega(x)}({}_zf)$. Therefore the map $f\to{}_zf$ induces a projection P with the desired properties.

Theorem 2. 2. *If $L_1(G)$ is equivalent to a C^*-algebra, then G is finite.*

Proof. Let P be the projection in Lemma 2.1. Then by [9, Thm. 2.9] there exists an additive set function φ on G such that $P(f)=\varphi * f$ for all f in $L_\infty(G)$. By Lemma 2.1, $L_\infty(G)$ consists entirely of continuous functions and so φ is regular, in the sense of [5, III. 5. 11]. Let V be a relatively compact open subset of G. Then ξ_V belongs to $C_0(G)$. So $\varphi(V)=\int\xi_V(y)\,d\varphi(y)=\check{\varphi}*\xi_V(e)=P(\xi_V)(e)=\xi_V(e)$. Therefore by the regularity of φ we have that $\varphi(\{e\})=1$ and $\varphi(\{x\})=0$ if $x\neq e$. By [9, Prop. 2.4 (ii)] the function $x\to\varphi(\{x\})$ belongs to $L_\infty(G)$ and so in our case it is continuous. This implies that $\{e\}$ is open in G and so G is discrete. But then since P is a projection, $c_0(G)$ is complemented in $l_\infty(G)$. This is impossible by [1, Cor. 2.2] unless G is finite.

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