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Conditions involving universally quantified function variables

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In his paper [1], W. TAYLOR gave a characterization for properties of varieties which are expressible by means of Mal'cev conditions. The goal of this note is to show that many natural and usual properties of varieties may be expressed by means of a sort of conditions, similar to the Mal'cev type ones.

To emphasize the analogy, we shall use the language of *heterogeneous clones* (i.e., of heterogeneous algebras of type τ_0) due to Taylor. A complete introduction to this subject may be found in [1], § 2, the knowledge of which will be supposed in the sequel. Especially, our notations are adopted from there. Note that letters x, y with exponent n stand for variables of type n ($n=1, 2, \ldots$; see p. 360 in [1]). As in [1], no use of nullary operations will be made.

We shall say that two (or more) sentences (in prenex normal form) are uniformly quantified if, neglecting variables as well as repetitions, they have the same quantifier symbol sequences.

The definition of Mal'cev condition may be formulated as follows (see [1], 2.16): Let $\mathcal{L}$ be a class of varieties. Suppose that there exists a countable sequence $\langle f_1, f_2, \ldots\rangle$ such that

(α^*) each f_i is the *existential* quantification of a (finite) conjunction of equations in the language of heterogeneous clones,

(β^*) for each $n, f_n \rightarrow f_{n+1}$ is true,

(γ^*) $\mathcal{V} \in \mathcal{L}$ if and only if for some n, $\mathfrak{A}(\mathcal{V}) \models f_n$, (i.e., f_n is true in the heterogeneous clone of polynomials of the free $\mathcal{V}$-algebra on a countable generating set). Then we say that $\mathcal{L}$ is defined by a Mal'cev condition.

We shall describe some classes of varieties which may be characterized by means of conditions containing *universal* quantifiers too. Several such ones were touched in [1], 6.7; here we give, essentially, a more complete list and a classification for them, emphasizing the request in [1], 6.8., to have characterizations for classes defined by conditions involving universal quantifiers in the language of heterogeneous clones.

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Theorem. *Let $\mathcal{K}_i$ be the class of all varieties $\mathcal{V}$ having the property* (i), *where*

(1) *in the algebras of $\mathcal{V}$ all congruence classes are subalgebras,*

(2) *in the algebras of $\mathcal{V}$ any operation* *) *applied to endomorphisms furnishes an endomorphism,*

(3) *every direct product $\mathbf{B}\times\mathbf{C}\in\mathcal{V}$ can be decomposed into a direct sum of its subalgebras $\mathbf{B}_1$ and $\mathbf{C}_1$ with $\mathbf{B}_1\cong\mathbf{B}$ and $\mathbf{C}_1\cong\mathbf{C}$,*

(4) *$\mathcal{V}$ is a variety of groups with multiple operators in the Higgins' sense,*

(5) *in the algebras of $\mathcal{V}$ among the classes of every congruence there is exactly one subalgebra,*

(6) *in the algebras of $\mathcal{V}$ all subalgebras are congruence classes,*

(7) *in the algebras of $\mathcal{V}$ any operation applied to subalgebras gives subalgebras,*

(8) *in the algebras of $\mathcal{V}$ any operation applied to classes of a congruence furnishes a (full) class of the same congruence,*

(9) *in the algebras of $\mathcal{V}$ every subalgebra is a class of a unique congruence and* (5) *holds,*

(10) *$\mathcal{V}$ satisfies* (5) *[whence it has an essentially nullary operation 0] and every direct product $\mathbf{B}\times\mathbf{C}\in\mathcal{V}$ is a free product in $\mathcal{V}$ of its subalgebras $\mathbf{B}\times\mathbf{0}$ and $\mathbf{0}\times\mathbf{C}$,*

(11) *in the algebras of $\mathcal{V}$ every subalgebra is a class of a unique congruence and* (1) *holds.*

Let $\mathcal{M}$ denote any fixed one of the classes $\mathcal{K}_i$ $(i=1, \ldots, 11)$. Then there exists a countable sequence $\langle f_1, f_2, \ldots\rangle$ such that

(α) *each f_i is a sentence in the language of heterogeneous clones whose matrix is a (finite) conjunction of equations, and all f_i are uniformly quantified,*

(β) *for each n, $f_{n+1}\to f_n$ is true,*

(γ) *$\mathcal{V}\in\mathcal{M}$ if and only if $\mathfrak{A}(\mathcal{V})\models f_n$ for all n.*

Proof. (1) means that in algebras of $\mathcal{V}$ each operation is idempotent, i.e., for any natural n, every n-ary operation is idempotent. This may be reformulated as $\mathfrak{A}(\mathcal{V})\models f_n$, where

$$f_n \equiv (\forall x^n)\,(C_n^n(x^n, e_1^n, \ldots, e_1^n) = e_1^n).$$

Clearly, the sentences f_i satisfy (α). Finally, suppose that $C_{n+1}^{n+1}(x^{n+1}, e_1^{n+1}, \ldots, e_1^{n+1}) = e_1^{n+1}$ holds identically in $\mathfrak{A}(\mathcal{V})$. Then, substituting $x^{n+1}=C_{n+1}^n(x^n, e_1^{n+1}, \ldots, e_n^{n+1})$ and using the identities of heterogeneous clones, we get $C_{n+1}^n(x^n, e_1^{n+1}, \ldots, e_1^{n+1})=e_1^{n+1}$ for any $x^n\in\mathfrak{A}(\mathcal{V})$, whence also f_n holds there, and thus (β) is fulfilled, too.

(2) means that in algebras of $\mathcal{V}$ any two operations commute (see, e.g., [7] and [2], p. 127). Clearly, this is equivalent to the requirement that, for any natural n,

*) Here and in what follows under operation we mean not only basic operations, but also polynomials of the considered algebra.

any two n-ary operations commute, i.e., $\mathfrak{A}(\mathscr{V})\models f_n$, where

$$f_n \equiv (\forall x^n)(\forall y^n)\big(C^n_{n^2}(x^n, C^n_{n^2}(y^n, e^{n^2}_1, \dots, e^{n^2}_n), C^n_{n^2}(y^n, e^{n^2}_{n+1}, \dots, e^{n^2}_{2n}), \dots$$

$$\dots, C^n_{n^2}(y^n, e^{n^2}_{n^2-n+1}, \dots, e^{n^2}_{n^2})) =$$

$$= C^n_{n^2}(y^n, C^n_{n^2}(x^n, e^{n^2}_1, e^{n^2}_{n+1}, \dots, e^{n^2}_{n^2-n+1}), \dots, C^n_{n^2}(x^n, e^{n^2}_n, e^{n^2}_{2n}, \dots, e^{n^2}_{n^2}))\big).$$

The remainder discussion may be performed as in the case of (1).

As follows from a result of J. Łoś ([3]; see also [1], 6.7), property (3) means that for any natural n, $\mathfrak{A}(\mathscr{V})\models f_n$, where

$$f_n \equiv (\exists x^1)(\exists x^2)(\forall y^n)\big(C^1_2(x^1, e^2_1) = C^1_2(x^1, e^2_2)\wedge C^2_1(x^2, e^1_1, C^1_1(x^1, e^1_1)) =$$

$$= C^2_1(x^2, C^1_1(x^1, e^1_1), e^1_1)\wedge C^n_1(y^n, C^1_1(x^1, e^1_1), \dots, C^1_1(x^1, e^1_1)) = C^1_1(x^1, e^1_1).$$

What concerns (4) the sentences defining groups with multiple operators can be written down in a similar fashion. Furthermore, (5) means the existence of an essentially nullary operation in $\mathscr{V}$, the result of which forms a subalgebra in every algebra of $\mathscr{V}$ (see [7], § 4). Equivalently, $\mathfrak{A}(\mathscr{V})\models f_n$ for any natural n, where

$$f_n \equiv (\exists x^1)(\forall y^n)\big(C^1_2(x^1, e^2_1) = C^1_2(x^1, e^2_2)\big)\wedge C^n_1(y^n, C^1_1(x^1, e^1_1), \dots, C^1_1(x^1, e^1_1)) =$$

$$= C^1_1(x_1, e^1_1).$$

(6) is called the Hamiltonian property and it is fulfilled if and only if for any n-ary ($n=1, 2, \dots$) operation g there exists a ternary operation h_g such that $g(x_1, \dots, x_n)=h_g(x_0, x_1, g(x_0, x_2, \dots, x_n))$ holds identically in $\mathscr{V}$ (see [4]). The assumption (7) means that for any n-ary ($n=1, 2, \dots$) operations g and h there exist n-ary operations $g_1, \dots, g_n$ such that $g(h(x_{11}, \dots, x_{1n}), \dots, h(x_{n1}, \dots, x_{nn}))= =h(g_1(x_{11}, \dots, x_{n1}), \dots, g_n(x_{1n}, \dots, x_{nn}))$ holds identically in $\mathscr{V}$ (cf. [7], p. 205). Furthermore, (8) is valid if and only if for any n-ary operation g there exist $n+1$-ary operations $g_1, \dots g_n$ such that $g_i(x_1, \dots, x_n, g(x_1, \dots, x_n))=x_i$ ($i=1, \dots, n$) and $g(g_1(x_1, \dots, x_n, x), \dots, g_n(x_1, \dots, x_n, x))=x$ hold identically in $\mathscr{V}$ (see [5], p. 242). We shall consider only the property (8); (6) and (7) can be treated analogously.

From the above form of condition (8), $\mathscr{V}$ has property (8) exactly then if for any natural n, $\mathfrak{A}(\mathscr{V})\models f_n$, where

$$f_n \equiv (\forall x^n)(\exists y^{n+1}_1)\dots(\exists y^{n+1}_n)\Big(\bigwedge_{i=1}^{n}(C^{n+1}_n(y^{n+1}_i, e^n_1, \dots, e^n_n, x^n) = e^n_i)\wedge$$

$$\wedge C^n_{n+1}(x^n, y^{n+1}_1, \dots, y^{n+1}_n) = e^{n+1}_{n+1}\Big).$$

We have to prove (β). Choose an arbitrary $x^n\in\mathfrak{A}(\mathscr{V})$. As f_{n+1} is valid, for $x^{n+1}= =C^n_{n+1}(x^n, e^{n+1}_1, \dots, e^{n+1}_n)$ there exist $y^{n+2}_1, \dots, y^{n+2}_{n+1}\in\mathfrak{A}(\mathscr{V})$, satisfying the matrix of

f_{n+1}. Then a routine computation shows that x^n and

$$y_i^{n+1} = \left(C_{n+1}^{n+2}(y_i^{n+2}, e_1^{n+1}, \dots, e_n^{n+1}, e_n^{n+1}, e_{n+1}^{n+1})\right)$$

satisfy the matrix of f_n, proving (β).

(9) means that $\mathcal{V}$ is equivalent to the variety of all unital modules over a ring with unit element (see [7], § 4). Furthermore, $\mathcal{V}$ is such a variety if and only if it has operations "addition" and "forming of inverse element" satisfying the axioms of Abelian groups, any operation in $\mathcal{V}$ commutes with the addition, and for any n-ary operation g there exist unary operations $g_1, \dots, g_n$ such that $g(x_1, \dots, x_n) = g_1(x_1) + \dots + g_n(x_n)$ holds identically in $\mathcal{V}$. (The easy verification of this fact may be omitted.) Obviously, these conditions can be rewritten in the form of a countable sequence of $\exists\forall\exists$ sentences f_n, satisfying $(\alpha)-(\gamma)$.

(10) means that $\mathcal{V}$ is equivalent to the variety of all unital modules over a semiring with unit element (see [6], Theorem 2). Now we can proceed as in the case (9), observing that varieties of unital modules over semirings may be characterized by the following properties: they have operations "addition" and "forming of neutral element" (a unary operation!) satisfying the axioms of Abelian monoids, any operation in $\mathcal{V}$ commutes with the addition, for any n-ary operation g there exist unary operations $g_1, \dots, g_n$ with the identity $g(x_1, \dots, x_n) = g_1(x_1) + \dots + g_n(x_n)$ in $\mathcal{V}$, and any unary operation is annihilated by the formation of neutral element.

As it was shown in [9], property (11) means that $\mathcal{V}$ is equivalent to the variety of all affine modules over a ring with unit element. Then a desired reformulation into the language of heterogeneous clones can be achieved using the following fact:

A variety $\mathcal{V}$ is equivalent to the variety of all affine modules over some ring **R** if and only if

(a) $\mathcal{V}$ has a ternary operation p commuting with itself and with every binary operation such that $p(x, y, x) = p(x, x, y) = y$ holds identically in $\mathcal{V}$,

(b) all binary operations are idempotent in $\mathcal{V}$,

(c) if $n \geqq 3$, for each n-ary operation g there exist binary operations $g_2, \dots, g_n$ such that

$$g(x_1, \dots, x_n) = p(x_1, \dots, p(x_1, p(x_1, g_2(x_1, x_2), g_3(x_1, x_3)), g_4(x_1, x_4)) \dots, g_n(x_1, x_n))$$

holds identically in $\mathcal{V}$.

To prove the necessity of (a)—(c), let us consider a ring **R** with unit element 1. In any affine **R**-module, take $p(x, y, z) = -x + y + z$ and for $g(x_1, \dots, x_n) = \gamma_1 x_1 + \dots + \gamma_n x_n$ $(\gamma_i \in \mathbf{R})$ let $g_k(x, y) = (1 - \gamma_k)x + \gamma_k y$ $(k = 2, \dots, n)$. Then (a)—(c) can be verified immediately.

Now assume the validity of (a)—(c). We have to prove that $\mathcal{V}$ is idempotent, regular and Hamiltonian (cf. Theorem 2 in [9]). The idempotency of operations in $\mathcal{V}$ follows from (a)—(c) directly. To prove the regularity it is enough to find a ternary operation r with the identity $r(x, x, z)=z$ and identical implication $r(x, y, z)=z \to x=y$ (see [8]). We show that $r=p$ is adequate. Indeed, $p(x, x, z)=z$ was assumed; let, on the other hand, $p(x, y, z)=z$. Then (a) implies

$$x = p(z, z, x) = p(z, p(x, y, z), x) = p(p(x, x, z), p(x, y, z)p(x, x, x)) =$$
$$= p(p(x, x, x), p(x, y, x), p(z, z, x)) = p(x, y, x) = y.$$

To prove that $\mathcal{V}$ is Hamiltonian, for any n-ary g a ternary h_g is needed with the identity $g(x_1, \dots, x_n)=h_g(x_0, x_1, g(x_0, x_2, \dots, x_n))$ in $\mathcal{V}$. We assert that for any at least binary g the operation $h_g(x_1, x_2, x_3)=p(x_1, g(x_2, x_1, \dots, x_1), x_3)$ is good. From (a) and (c) it follows that p commutes with each operation; hence we get

$$h_g(x_0, x_1, g(x_0, x_2, \dots, x_n)) = p(g(x_0, \dots, x_0), g(x_1, x_0, \dots, x_0), g(x_0, x_2, \dots, x_n)) =$$
$$= g(p(x_0, x_1, x_0), p(x_0, x_0, x_2), \dots, p(x_0, x_0, x_n)) = g(x_1, x_2, \dots, x_n).$$

This completes the discussion of the case (11) and also the proof of our Proposition.

Remark that properties (1) and (2) are defined with the aid of $\forall$ sentences, (3)—(5) involve $\exists\forall$ sentences, (6)—(8) require $\forall\exists$ sentences and (9)—(11) may be expressed by $\exists\forall\exists$ sentences.

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