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# Extensions of partial multiplications and polynomial identities on Abelian groups

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(i) In this paper $G$ will denote an abelian group, and $A$ will denote a subgroup of $G$. A multiplication on $A$ is meant to be a homomorphism $\mu\colon A\times A\to A$, and a partial multiplication on $A$ is meant to be a homomorphism $\mu\colon A\times A\to G$ [1, vol. II, pp. 281—284]. A multiplication $\varphi$ on $G$ is called an extension of a partial multiplication $\mu$ on $A$ if the restriction of $\varphi$ to $A$, $\varphi|_A=\mu$. In (ii) conditions are given for which every partial multiplication on $A$ extends to a multiplication on $G$.

$P(X_1, \ldots, X_n)$ will denote a polynomial in non-commuting variables over the ring of integers. A partial multiplication $\mu$ on $A$ is said to satisfy a polynomial identity $P(X_1, \ldots, X_n)$ if the elements of $(A, \mu)$ satisfy $P(X_1, \ldots X_n)=0$. In (iii) conditions are given for which a multiplication on $G$ extending a partial multiplication $\mu$ on $A$ satisfies polynomial indentities statisfied by $\mu$. Polynomial identities which a multiplication on a torsion free group can satisfy are examined in (iv).

(ii) Theorem 1. *Let $A$ be a torsion free subgroup of $G$. Every partial multiplication on $A$ can be extended to a multiplication on $G$ under each of the following conditions:*

1. *$G$ is divisible,*
2. *$(G\otimes G)/(A\otimes A)$ is free,*
3. *$(G\otimes G)/(A\otimes A)$ is a torsion group, and $G$ is $p$-divisible for every prime $p$ for which $(G\otimes G)/(A\otimes A)$ has a non-trivial $p$-component.*

Proof. The sequence

$$0 \to A\otimes A \to G\otimes G \to (G\otimes G)/(A\otimes A) \to 0$$

is exact [3, Theorem 2.8]. Therefore, the sequence

$$\operatorname{Hom}(G\otimes G, G) \to \operatorname{Hom}(A\otimes A, G) \xrightarrow{\varphi} \operatorname{Ext}\big((G\otimes G)/(A\otimes A), G\big)$$

is exact. Each of the conditions 1—3 assures that $\operatorname{Ext}\big((G\otimes G)/(A\otimes A), G\big)=0$, so that $\varphi$ is an epimorphism.

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(iii) Theorem 2. *Let G be torsion free, and let A be an essential subgroup of G (i.e. $G/A$ is a torsion group). Let $\mu$ be a partial multiplication on A satisfying a homogeneous polynomial identity $P(X_1, \dots, X_n)$. If $\bar{\mu}$ is a multiplication on G which extends $\mu$, then $\bar{\mu}$ satisfies $P(X_1, \dots, X_n)$.*

Proof. Let $m = \deg P(X_1, \dots, X_n)$, and let $g_1, \dots, g_n \in G$. There exist positive integers $l_i$ such that $l_i g_i \in A$, $1 \leqq i \leqq n$. Let $l = \prod_{i=1}^{n} l_i$. Then $lg_i \in A$, $1 \leqq i \leqq n$. Therefore $0 = P(lg_1, \dots, lg_n) = l^m P(g_1, \dots, g_n)$. $G$ is torsion free, so that $P(g_1, \dots, g_n) = 0$.

Corollary 1. *Let G be a torsion free group, and let B be an A-high subgroup of G. Let $\mu$ be a partial multiplication on $A \otimes B$, and let $\mu_A$ and $\mu_B$ respectively be the restrictions of $\mu$ to A and to B. Let $\mu_A$ and $\mu_B$ satisfy a homogeneous polynomial $P(X_1, \dots, X_n)$. Then* 1. *$\mu$ satisfies $P(X_1, \dots, X_n)$, and* 2. *every multiplication $\bar{\mu}$ on G which extends $\mu$ satisfies $P(X_1, \dots, X_n)$.*

Proof. The homogeneity of $P(X_1, \dots, X_n)$ clearly implies 1. Let $\bar{\mu}$ be a multiplication on $G$ which extends $\mu$. $G/(A \otimes B)$ is a torsion group [1, vol I, p. 50 ex. 6]. By Theorem 2, $\bar{\mu}$ satisfies $P(X_1, \dots, X_n)$.

Corollary 2. *For every positive integer $n \geqq 2$ there exists a nilpotent ring R with degree of nilpotency n such that the additive group G of R satisfies:*

1. *G is divisible and torsion free.*
2. *G is the divisible hull of a group A whose nilstufe* [4] *is $n-1$.*

Proof. Szele [4, Theorem 2] has shown that there exists a torsion free group $A$ with nilstufe $n-1$. Let $\mu$ be a multiplication on $A$ for which $A^{n-1} \neq 0$. $\mu$ satisfies $P(X_1, \dots, X_n) = X_1 X_2, \dots, X_n$. Let $G$ be the divisible hull of $A$. By Theorem 1, $\mu$ can be extended to a multiplication on $G$, and by Theorem 2, $\bar{\mu}$ satisfies $P(X_1, \dots, X_n)$.

Theorem 3. *Let A be an essential subgroup of G, and let $\mu$ be a partial multiplication on A such that $(A, \mu)$ does not possess any nonzero left zero divisors. Then for any multiplication $\bar{\mu}$ on G extending $\mu$, the nonzero elements of A are not left zero divisors in $(G, \bar{\mu})$.*

Proof. Let $0 \neq a \in A$. Define $\varphi_a \colon A \to G$, $\varphi_a(A') = \mu(a, a')$ for all $a' \in A$. $\varphi_a$ is a homomorphism on $A$. Since $a$ is not a left zero divisor in $(A, \mu)$, $\varphi_a$ is a monomorphism. Let $\bar{\mu}$ be a multiplication on $G$ extending $\mu$. Define $\bar{\varphi}_a \colon G \to G$, $\bar{\varphi}_a(g) = \mu(a, g)$ for all $g \in G$. $\bar{\varphi}_a$ is an endomorphism of $G$, with the restriction of $\bar{\varphi}_a$ to $A$, $\bar{\varphi}_a|_A = \varphi_a$. By [1, Lemma 24.2] $\bar{\varphi}_a$ is a monomorphism. Hence a is not a left zero divisor in $(G, \bar{\mu})$.

(iv) Theorem 4. *Let G be a torsion free group, and let $\mu$ be a multiplication on G satisfying a homogeneous polynomial $P(X_1, \dots, X_n)$ of degree r. Let C be the sum of the coefficients of $P(X_1, \dots, X_n)$. Then either $\mu$ satisfies $X^r$, or $C = 0$.*

Proof. Let $0 \neq g \in G$. Clearly, $0 = P(g, \dots, g) = Cg^r$. $G$ is torsion free. Therefore, if $C \neq 0$, then $g^r = 0$.

Theorem 5. *Let $R$ be a ring satisfying the polynomial identity*

$$P(X_1, X_2) = aX_1^2 + bX_2^2 + CX_1X_2 + dX_2X_1 + eX_1 + fX_2.$$

*Then $R$ satisfies $b(XY+YX)$.*

Proof. If $R$ satisfies $P(X_1, X_2)$, then $R$ satisfies

$$P_1(X_1, X_2, X_3) = P(X_1+X_3, X_2) - P(X_1, X_2) - P(X_3, X_2) = \\ = a(X_1X_3 + X_3X_1) - bX_2^2 - fX_2.$$

$R$ also satisfies

$$P_2(X_1, X_2, X_3, X_4) = P_1(X_1+X_4, X_2, X_3) - P(X_1, X_2, X_3) - P_1(X_4, X_2, X_3) = \\ = bX_2^2 + fX_2,$$

or $P_2(X) = bX^2 + fX$. This implies that $R$ satisfies

$$P_3(X, Y) = P_2(X+Y) - P_2(X) - P_2(Y) = b(XY+YX).$$

The following are direct consequences of Theorem 5 or its proof:

Corollary 1. *Let $G$ be a torsion free group, and let $\mu$ be a multiplication on $G$ satisfying $P(X_1, X_2)$ of theorem 5 with $b \neq 0$. Then $\mu$ satisfies $XY+YX$. If $\mu$ is commutative, then $\mu$ satisfies $XY$.*

Corollary 2. *Let $R$ be a commutative ring satisfying $P(X_1, X_2)$ of Theorem 5. Let $\pi$ be the set of prime divisors of $b$ and let $\pi'$ be the set of primes $p$ for which the additive group of $R$ has a nonzero $p$-primary component. If $\pi \cap \pi' = \emptyset$, then $R$ satisfies $XY$.*

Corollary 3. *Let $R$ be a ring satisfying $P(X_1, X_2)$ of Theorem 5 with $b \neq 0$. Then for every $a \in R$, $\{a, a^2\}$ is a dependent set* [1 vol. I, p. 83].

Corollary 4. *Let $R$ be a ring satisfying $P(X_1, X_2)$ of Theorem 5, with $b=0$, $f \neq 0$. Then the additive group of $R$ is bounded.*

Theorem 6. *Let $G$ be a torsion free group of finite rank $n$ such that for every $0 \neq g \in G$, the type of $g$, $T(g)$, is not idempotent. Then every multiplication on $G$ satisfies $X^{2^n}$.*

Proof. Koehler [2, Theorem 1.6] has shown that every ascending chain of types realizable in $G$, $t_1 < t_2, < \dots < t_r$, with $t_r \neq (\infty, \dots, \infty, \dots)$ is of length less than or equal to $n$. Let $0 \neq g \in G$. For every multiplication on $G$

$$(*) \qquad T(g) \leqq T(g^2) \leqq T(g^4) \leqq \dots \leqq T(g^{2^n}).$$

Suppose that $T(g^{2^{k+1}})=T(g^{2^k})$, $T(g^{2^{k+1}})\geqq 2T(g^{2^k})$ for some $0\leqq k\leqq n$ so that $T(g^{2^k})$ is idempotent, and hence $g^{2^k}=0$. If $T(g^{2^k})<T(g^{2^{k+1}})$ for all $k$, $0\leqq k<n$, then $(*)$ is a chain of length $n+1$, and hence $T(g^{2^n})=(\infty, \ldots, \infty, \ldots)$ which implies that $g^{2^n}=0$.

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