Acta Sci. Math., 38 (1976), 21—43

On products of abstract automata

F. GÉCSEG

Frequently two automata behave exactly in the same way as far as the transitions induced by their inputs are concerned, but none of them can be represented homomorphically by a (general) power of the other one; although the existence of homomorphisms between automata does not imply that they have common input sets. This situation can be avoided by allowing input words as input signals of the component automata. This modification leads to the concept of a generalized product introduced in this paper. Furthermore, we allow input words as counter images of input signals under homomorphic representations. The resulting representations will be called simulations.

The purpose of this paper is to study the generalized products and simulations from the point of view of isomorphic and homomorphic completeness. It will turn out that in most cases the generalized products and simulations are more effective than the classical products and representations. Furthermore, the results concerning generalized products and simulations will be interpreted in terms of classical products, representations and temporal products of automata.

By an *automaton* we mean a triplet $\mathbf{A}=(X, A, \delta)$, where X and A are nonvoid finite sets called the *input set* and *state set*, respectively. Moreover, $\delta: A\times X\to A$ denotes the *transition function* of $\mathbf{A}$.

Take an arbitrary finite group G, and form the automaton $\mathbf{G}=(G, G, \delta_{\mathbf{G}})$ with $\delta_{\mathbf{G}}(g_1, g_2)=g_1g_2$ for all $g_1, g_2\in G$, where g_1g_2 means that g_1 is multiplied by g_2 in G. $\mathbf{G}$ is a *grouplike automaton*.

For any nonvoid set X, let us denote by $F(X)$ the free monoid generated by X. If X is an input set of an automaton $\mathbf{A}=(X, A, \delta)$ then the elements $p\in F(X)$ are called *input words* of A. The transition function δ can be extended to $A\times F(X)\to A$ in a natural way: for any $p=p'x\in F(X)$ and $a\in A$, $\delta(a, p)=\delta(\delta(a, p'), x)$. Further on we shall use the more convenient notation $ap_{\mathbf{A}}$ for $\delta(a, p)$. If there is no danger of confusion then we omit the index $\mathbf{A}$.

Received December 18, 1974.

Let $\mathbf{A}=(X, A, \delta)$ be an automaton. Define a binary relation $\varrho_{\mathbf{A}}$ on $F(X)$ in the following manner: for two input words $p, q \in F(X), p \equiv q(\varrho_{\mathbf{A}})$ if and only if $ap_{\mathbf{A}}=aq_{\mathbf{A}}$ for all $a \in A$. The quotient semigroup $F(X)/\varrho_{\mathbf{A}}$ is called the *characteristic semigroup* of $\mathbf{A}$, and it will be denoted by $S(\mathbf{A})$. We use the notation $[p]_{\mathbf{A}}$ for the element of $S(\mathbf{A})$, containing $p \in F(X)$. Thus, $[p]_{\mathbf{A}}=[q]_{\mathbf{A}}$ $(p, q \in F(X))$ if and only if p and q induce the same transition in A. Again, if there is no danger of confusion, we omit the index $\mathbf{A}$ in $[p]_{\mathbf{A}}$.

Take an automaton $\mathbf{A}=(X, A, \delta)$, and let π be a partition of A. It is said that π has the *substitution property* (shortly, SP) if $a \equiv b(\pi)$ implies $\delta(a, x) \equiv \delta(b, x)(\pi)$ for all $a, b \in A$ and $x \in X$. (Let us note that we use the same symbol π for a partition and for the equivalence relation inducing it.) The quotient automaton induced by π will be denoted by $\mathbf{A}/\pi$.

Let $\mathbf{A}_i=(X_i, A_i, \delta_i)$ $(i=1, \ldots, n)$ be a system of automata. Moreover, let X be a finite nonvoid set, and φ a mapping of $A_1 \times \ldots \times A_n \times X$ into $F(X_1) \times \ldots \times F(X_n)$. We say that the automaton $\mathbf{A}=(X, A, \delta)$ with $A=A_1 \times \ldots \times A_n$ and

$$\delta((a_1, \ldots, a_n), x) = (a_1 p_1, \ldots, a_n p_n),$$

where $(p_1, \ldots, p_n)=\varphi(a_1, \ldots, a_n, x)$, is the *generalized product* of $\mathbf{A}_i$ $(i=1, \ldots, n)$ with respect to X and φ. For this product we use the shorter notation $\mathbf{A}=\prod_{i=1}^{n} \mathbf{A}_i[X, \varphi]$.

A generalized product $\mathbf{A}=\prod_{i=1}^{n} \mathbf{A}_i[X, \varphi]$ is a *generalized α_i-product* $(i=0, 1, \ldots)$ if φ can be given in the form

$$\varphi(a_1, \ldots, a_n, x) = (\varphi_1(a_1, \ldots, a_n, x), \ldots, \varphi_n(a_1, \ldots, a_n, x))$$

such that each φ_j $(1 \leqq j \leqq n)$ is independent of states having indices greater than or equal to $j+i$.

If in a generalized product [generalized α_i-product] φ is of the form φ: $A_1 \times \times \ldots \times A_n \times X \to X_1 \times \ldots \times X_n$ then we get the concept of a *product* [α_i-product] (see [3]). Moreover, if in a generalized product [product] $\mathbf{A}$, $\mathbf{A}_i=\mathbf{B}$ for all $i(=1, \ldots, n)$ then $\mathbf{A}$ is called a *generalized power [power]* of $\mathbf{B}$.

The concept of the generalized α_i-product (α_i-product) can be interpreted in the following way. For a given generalized product (product) take a well ordering on the set of its components. Assume that $\mathbf{A}_i$ is the i-th automaton under this well ordering. If for two j and i with $i \leqq j$ there is a feed-back from $\mathbf{A}_j$ to A_i then we say that the length of this feed-back is $j-i+1$. Now for any $i(=0, 1, \ldots)$, in the generalized α_i-products (α_i-products) the lengths of such feed-backs does not exceed i under the usual well ordering of natural numbers.

We say that an automaton $\mathbf{A}=(X, A, \delta)$ *homomorphically simulates* $\mathbf{B}=(X', B, \delta')$ if there exist a one-to-one mapping τ_1 of X' into $F(X)$ and a mapping τ_2 of a subset A' of A onto B such that $\tau_2(a\tau_1(x'))=\delta'(\tau_2(a), x')$ for any $a \in A'$ and $x' \in X'$. If τ_2 is one-to-one as well then we speak of an *isomorphic simulation.* Furthermore, if τ_1 is of the form $\tau_1: X' \to X$, then we speak of *homomorphic* and *isomorphic representations.*

The following result is trivial.

Lemma 1. *If* **A** *homomorphically simulates* **B** *and* **B** *homomorphically simulates* **C**, *then* **C** *can be simulated homomorphically by* **A**. *Similar statement is valid for isomorphic simulations.*

A system $\sum$ of automata is called *homomorphically S-complete* with respect to the generalized product [generalized α_i-product] if any automaton can be simulated homomorphically by a generalized product [generalized α_i-product] of automata from $\sum$. The concept of *isomorphic S-completeness* is defined similarly.

Take a system $\sum$ of automata. For any $\mathbf{A}=(X, A, \delta)\in\sum$ denote by $\mathbf{A}^*=(X^*, A, \delta^*)$ the automaton whose input set X^* is $S(\mathbf{A})$ and $\delta^*(a, [p])=ap_{\mathbf{A}}$ $([p]\in S(\mathbf{A}))$.

The following statement is obvious.

Lemma 2. *For every generalized product (generalized α_i-product)* $\mathbf{B}=\prod_{i=1}^{n}\mathbf{B}_i[X, \varphi]$ *there is a product (α_i-product)* $\mathbf{B}'=\prod_{i=1}^{n}\mathbf{B}_i^*[X, \varphi^*]$ *such that* **B** *is isomorphic to* **B**′, *and conversely.*

Now we are ready for studying isomorphic and homomorphic S-completeness with respect to different types of generalized products.

1. α_0-products

For any natural number n, denote by $\mathbf{T}_n=(T_n, N, \delta_N)$ the automaton for which $N=\{1, \ldots, n\}$, T_n is the set of all transformations t of N, and $\delta_N(j, t)=t(j)$ for all $j\in N$ and $t\in T_n$.

Theorem 1. *A system $\sum$ of automata is isomorphically S-complete with respect to the generalized α_0-product if and only if for any natural number n, there exists an automaton* $\mathbf{B}\in\sum$ *such that* **B** *isomorphically simulates* $\mathbf{T}_n$.

Proof. In order to prove the sufficiency of these conditions take an automaton $\mathbf{A}=(X, A, \delta)$ with n states. Let τ_2 be an arbitrary 1—1 mapping of A onto $N=\{1, \ldots, n\}$. Form the α_0-product $\mathbf{T}_n'=(\mathbf{T}_n)[X, \varphi]$, where $\varphi(x)=t$ $(x\in X, t\in T_n)$ such that $\tau_2(\delta(a, x))=t(\tau_2(a))$ for any $a\in A$. Let τ_1 denote the identity mapping on X. Then (τ_1, τ_2^{-1}) gives an isomorphic simulation of **A** by an α_0-product of $\mathbf{T}_n$. Moreover, by our assumption, there exists an automaton **B** in $\sum$ which isomorphically simulates $\mathbf{T}_n$. Therefore, by Lemma 1, **A** can be simulated isomorphically by a generalized α_0-power of **B**.

Conversely, let $n>1$ be a natural number, and take $\mathbf{T}_n$. Assume that a generalized α_0-product $\mathbf{B}=(X, B, \delta')=\prod_{i=1}^{k} \mathbf{B}_i[X, \varphi]$ of automata from Σ isomorphically simulates $\mathbf{T}_n$. Then, by Lemma 2, $\mathbf{T}_n$ can be simulated isomorphically by an α_0-product $\mathbf{B}'=(X, B, \delta'')=\prod_{i=1}^{k} \mathbf{B}_i^*[X, \varphi^*]$, under two mappings τ_1: $T_n \to F(X)$ and τ_2: $B' \to N$ $(B' \subseteqq B)$.

The elements b of B can be written in the vectorial form $b=(b_1, \ldots, b_k)$ $(b_j \in B_j$ and B_j is the state set of $\mathbf{B}_j^*)$. Define partitions π_j' $(j=1, \ldots, k)$ on B in the following way:

$$b(=(b_1, \ldots, b_k)) \equiv ((b_1', \ldots, b_k')=)b'(\pi_j') \quad (b, b' \in B)$$

if and only if $b_1=b_1', \ldots, b_j=b_j'$. Now let π_j $(j=1, \ldots, k)$ be partitions on N given as follows: for any $b, b' \in B'$ we have $\tau_2(b) \equiv \tau_2(b')$ (π_j) if and only if $b \equiv b'$ (π_j'). It is easy to prove that the partitions π_j have SP.

On the other hand, on $\mathbf{T}_n$ only the two trivial partitions have SP. Thus, we get that each π_j has one-element blocks only, or it has one block only. Among these partitions there should be at least one which has more than one block, since $n>1$. Let l be the least index for which π_l has at least two blocks. Then the blocks of π_l consist of single elements. Therefore, the number of all blocks of π_l is n. We show that $\mathbf{B}_l^*$ isomorphically simulates $\mathbf{T}_n$.

By our assumption and the definition of π_j, all elements of B' coincide in their first $l-1$ components; let us denote them by $b_1', \ldots, b_{l-1}'$. Moreover, denote by B_l' the set of all l-th components of elements from B', and let X_l^* be the input set of $\mathbf{B}_l^*$. Define two mappings τ_1': $T_n \to F(X_l^*)$ and τ_2': $B_l' \to A$ in the following way: if $\tau_1(t)=x^{(1)} \ldots x^{(u)}$ then let

$$\tau_1'(t) = \varphi_l^*(b_1', \ldots, b_{l-1}', b_l, \ldots, b_k, x^{(1)}) \ldots$$
$$\ldots \varphi_l^*((b_1', \ldots, b_{l-1}', b_l, \ldots, b_k)(x^{(1)} \ldots x^{(u-1)})_{\mathbf{B}'}), (x^{(u)}),$$

and if $\tau_2(b)=a$ $(b \in B', a \in N)$ and b_l is the l-th component of b then let $\tau_2'(b_l)=a$. (Note that, by the definition of the α_0-product, φ_l^* is independent of states having indices greater than or equal to l.) It is obvious that τ_2' is a one-to-one mapping of B_l' onto N. Let us take a $b_l' \in B_l'$ and a $t \in T_n$. Then there exits a $b \in B'$ with $b=(b_1', \ldots, b_{l-1}', b_l', b_{l+1}, \ldots, b_k)$ such that $\tau_2(b)=\tau_2'(b_l')=a$. Therefore, if $\tau_1(t)= =x^{(1)} \ldots x^{(u)}$ then

$$b\tau_1(t) = (b_1', \ldots, b_{l-1}', b_l' \varphi_l^*(b_1', \ldots, b_{l-1}', b_l', b_{l+1}, \ldots, b_k, x^{(1)}) \ldots$$
$$\ldots \varphi_l^*((b_1', \ldots, b_{l-1}', b_l', b_{l+1}, \ldots, b_k)(x^{(1)} \ldots x^{(u-1)})_{\mathbf{B}'}, x^{(u)}), \ldots),$$

since

$$b_v' \varphi_v^*(b_1', \ldots, b_l', b_{l+1}, \ldots, b_k, x^{(1)}) \ldots$$
$$\ldots \varphi_v^*((b_1', \ldots, b_l', b_{l+1}, \ldots, b_k)(x^{(1)} \ldots x^{(u-1)})_{\mathbf{B}'}, x^{(u)}) = b_v'$$

for any $v<l$. From this we get that the l-th component of $b\tau_1(t)$ is $b_l'\tau_1'(t)$, showing that $\tau_2'(b_l'\tau_1'(t))=\delta_N(\tau_2'(b_l'), t)$. Since τ_2' is 1—1, thus $\mathbf{B}_l\in\sum$ isomorphically simulates $\mathbf{T}_n$.

The case $n=1$ can be proved by a similar argument.

From Theorem 1 we get the following

Corollary. *There exists no system of automata which is isomorphically S-complete with respect to the generalized α_0-product and minimal.*

Proof. Take a system $\sum$ of automata which is isomorphically S-complete with respect to the generalized α_0-product. Moreover, let $\mathbf{A}\in\sum$ be an automaton with n states, and take a natural number $m>n$. It is obvious that $\mathbf{A}$ is isomorphic to a subautomaton of an α_0-product of $\mathbf{T}_m$ (having one factor only). Furthermore, by Theorem 1, there exists a $\mathbf{B}\in\sum$ which isomorphically simulates $\mathbf{T}_m$. Therefore, $\mathbf{A}$ can be simulated isomorphically by a generalized α_0-power of $\mathbf{B}$. Thus, $\sum$-$\{\mathbf{A}\}$ is isomorphically S-complete with respect to the generalized α_0-product, showing that $\sum$ is not minimal.

Take the automaton $\mathbf{A}=(X, A, \delta)$ with $X=\{x, y, z\}$, $A=\{a_1, a_2\}$ and $\delta(a_1, x)=$ $=\delta(a_2, x)=\delta(a_2, z)=a_2$ and $\delta(a_2, y)=\delta(a_1, y)=\delta(a_1, z)=a_1$. This $\mathbf{A}$ is called a *two-state reset automaton*. Let us denote by H_2 the characteristic semigroup of $\mathbf{A}$.

For homomorphic simulations we have

Theorem 2. *A system $\sum$ of automata is homomorphically S-complete with respect to the generalized α_0-product if and only if the following conditions are satisfied:*

(i) *For any simple group G there exists a $\mathbf{B}\in\sum$ such that G is a homomorphic image of a subgroup of $S(\mathbf{B})$;*

(ii) *There exists $\mathbf{C}\in\sum$ such that H_2 is a homomorphic image of a subsemigroup of $S(\mathbf{C})$.*

Proof. The necessity of these conditions follows from the well known theorem of Krohn and Rhodes. (For a nice presentation of the Krohn—Rhodes theory, see [6].)

To prove the sufficiency of (i) and (ii), again, by the Krohn—Rhodes theorem, it is enough to show that: Every grouplike automaton $\mathbf{G}=(G, G, \delta_G)$ with a simple group G ($|G|>1$) and a two-state reset automaton can be given as a homomorphic image of a subautomaton of an α_0-product $\prod_{i=1}^{k}\mathbf{B}_i^*[X, \varphi^*]$, where $\mathbf{B}_i\in\sum$.

Take a grouplike automaton $\mathbf{G}=(G, G, \delta_G)$, where G ($|G|>1$) is a simple group. By condition (i), there exists a $\mathbf{B}\in\sum$ such that G is a homomorphic image of a subgroup G' of $S(\mathbf{B})$, under a homomorphism $\tau: G'\to G$. Let $\mathbf{B}$ be given in the form $\mathbf{B}=(X, B, \delta)$. Now define an α_0-product $\mathbf{B}'=(\mathbf{B}^*)[G, \varphi^*]$, where φ^* is an isomorphism of $F(G)$ into $F(G')$ such that $\tau(\varphi^*(g))=g$ for any $g\in G$. Take an arbitrary identity $up=vq$ over G, where u, v are variables and $p, q\in F(G)$. Assume that this identity holds on $\mathbf{B}'$. Since $S(\mathbf{B}')$ is a group (isomorphic to a subgroup of G'), thus there exists a subset B' of B such that each element of G induces a permutation of B' (in $\mathbf{B}'$), and distinct elements of G induce distinct permutations. It is obvious that $|B'|>1$. The identity $up=vq$ implies $up=vp$. But p induces a permutation of B'. Therefore, for any two elements a and b of B', we have $ap\neq bp$ if $a\neq b$. Thus, all identities holding on $\mathbf{B}'$ should have the form $up=uq$, i.e., $[\varphi^*(p)]=[\varphi^*(q)]$ in $S(\mathbf{B})$ whenever $up=uq$ holds in $\mathbf{B}'$. Now, by the choice of φ^*, $p=\tau(\varphi^*(p))=\tau(\varphi^*(q))=q$, i.e., $up=uq$ holds in $\mathbf{G}$. Therefore, we got that $\mathbf{G}$ is contained in the equational class generated by $\mathbf{B}'$. Thus, by the Theorem in [2], $\mathbf{G}$ is a homomorphic image of a subautomaton of a finite direct power of $\mathbf{B}'$. Since the direct product is a special case of the α_0-product, thus $\mathbf{G}$ is a homomorphic image of a subautomaton of an α_0-power of $\mathbf{B}'$. Consequently, by Lemma 2, $\mathbf{G}$ can be simulated homomorphically by a generalized α_0-power of $\mathbf{B}$.

Finally, if (ii) holds, then $\mathbf{C}^*$ has a subautomaton which is a two-state reset automaton (see [6], p. 148). This completes the proof of Theorem 2.

Since for any simple group G with n elements there exists a simple group G' with $|G'|>n$ such that G is isomorphic to a subgroup of G', thus from Theorem 2 we get

Corollary 1. *There exists no system of automata which is homomorphically S-complete with respect to the generalized α_0-product and minimal.*

Moreover, Theorems 1 and 2 imply

Corollary 2. *There exists a system $\sum$ of automata such that $\sum$ is homomorphically S-complete with respect to the generalized α_0-product and $\sum$ is not isomorphically S-complete with respect to the generalized α_0-product.*

2. α_1-products

We start with the study of homomorphic S-completeness with respect to the generalized α_1-products.

Theorem 3. *A system $\sum$ of automata is homomorphically S-complete with respect to the generalized α_1-product if and only if for any natural number n, there exist an automaton $\mathbf{A}=(X, A, \delta)$ in $\sum$, states $a_1, \dots, a_n \in A$ and input words $p_{jl}\in F(X)$ $(1\leqq j, l\leqq n)$ such that $a_j p_{jl}=a_l$.*

Proof. Let $\sum$ be a system of automata which is homomorphically S-complete with respect to the generalized α_1-product. Let n be a natural number, and take a prime $r>n$. Define an automaton $\mathbf{A}_r=(X', A_r, \delta_r)$ in the following way: $X'=\{x\}$,

$A_r=\{a_0, \ldots, a_{r-1}\}$ and

$$\delta_r(a_i, x) = \begin{cases} a_{i+1} & \text{if} \quad i < r-1, \\ a_0 & \text{if} \quad i = r-1. \end{cases}$$

Assume that $\mathbf{A}_r$ can be simulated homomorphically by a generalized α_1-product $\mathbf{B}=\prod_{i=1}^{k} \mathbf{B}_i[\bar{X}, \varphi]$ of automata from $\sum$. Thus, by Lemma 2, there exists an α_1-product $\mathbf{B}'=(\bar{X}, B, \delta')=\prod_{i=1}^{k} \mathbf{B}_i^*[\bar{X}, \varphi^*]$ which homomorphically simulates $\mathbf{A}_r$ under a set $B' \subseteq B$ and mappings $\tau_1(x)=p\in F(\bar{X})$ and $\tau_2: B' \to A_r$.

Let us represent the elements of B in the vectorial form $b=(b_1, \ldots, b_k)$. Define the partitions π_j' $(j=1, \ldots, k)$ on B in the same way as in the proof of Theorem 1. It can be shown by a short computation that these partitions π_j' have SP.

By the choice of $\mathbf{A}_r$, there exists a subset $B''=\{b_0', \ldots, b_{u-1}'\}$ of B' such that $r|u$,

$$b_l' \, p_{\mathbf{B}'} = \begin{cases} b_{l+1}' & \text{if} \quad l < u-1, \\ b_0' & \text{if} \quad l = u-1. \end{cases}$$

and $\tau_2(b_l')=a_{l(\bmod r)}$, where $l(\bmod r)$ denotes the least nonnegative residue of l modulo r. Let π_j be the restriction of π_j' to B''. It can be proved that for any j, the blocks of π_j have the same cardinality. Donete by f_1 the number of blocks of π_1. Moreover, it is easy to show that $\pi_1 \geqq \pi_2 \geqq \ldots \geqq \pi_k$, and each block of π_j contains the same number f_{j+1} of blocks of π_{j+1} $(j=1, \ldots, k-1)$. Therefore, $u=f_1 f_2 \ldots f_k$. But $r|u$ and r is a prime. Thus, there exists an $l(1\leqq l\leqq k)$ such that $r|f_l$. This means, by the definition of the partitions π_j, that the number of states of $\mathbf{B}_l^*$ occuring as l-th components in the elements of B'' is at least $f_j \geqq r$. Let us denote them by $c_1, \ldots, c_s$. Since for any two elements b' and b'' of B'' there exists an input word $q=p\ldots p$ such that $b' q_{\mathbf{B}'}=b''$, thus for any c_t, c_h $(1\leqq t, h \leqq s)$ there is an input signal x_{th} of $\mathbf{B}_l^*$ with $c_t x_{th}=c_h$ in $\mathbf{B}_l^*$. Consequently, by the definition of $\mathbf{B}_l^*$, $\mathbf{B}_l \in \sum$ satisfies the conditions of Theorem 3.

Conversely, assume that the conditions of Theorem 3 are satisfied. Take an arbitrary automaton $\mathbf{B}=(X, B, \delta_{\mathbf{B}})$ with $B=\{b_1, \ldots, b_n\}$. Then there exist an automaton $\mathbf{A}=(\bar{X}, A, \delta_{\mathbf{A}})\in \sum$, states $a_1, \ldots, a_n \in A$ and input signals x_{ij} $(1\leqq i, j\leqq n)$ of $\mathbf{A}^*$ such that $\delta_{\mathbf{A}}^*(a_i, x_{ij})=a_j$. Now take the α_1-product $\mathbf{C}=(X, C, \delta_{\mathbf{C}})=(\mathbf{A}^*)[X, \varphi^*]$, where for any $x\in X$, $\varphi^*(a_i, x)=x_{ij}$ if $\delta_{\mathbf{B}}(b_i, x)=b_j$ $(i, j=1, \ldots, n)$, and in all other cases $\varphi^*(a, x)(a\in A)$ is defined arbitrarily. It is obvoius that $\mathbf{C}$ *isomorphically* simulates $\mathbf{B}$.

From the above proof we get

Corollary 1. *A system of automata is homomorphically S-complete with respect to the generalized α_1-product if and only if it is isomorphically S-complete with respect to the generalized α_1-product.*

Corollary 2. *There exists no system of automata which is homomorphically (or isomorphically) S-complete with respect to the generalized α_1-product and minimal.*

The following result shows that the homomorphic and isomorphic simulations with respect to the generalized α_1-product do not coincide if they are considered over an arbitrary system of automata.

Theorem 4. *There exist a system $\sum$ of automata and an automaton* **A** *such that* **A** *can be simulated homomorphically by a generalized α_1-product of automata from $\sum$ and* **A** *cannot be simulated isomorphically by any generalized α_1-product of automata from $\sum$.*

Proof. Take the following automaton $\mathbf{A}=(X, A, \delta)$, where $X=\{x, y\}$, $A=\{a, b, c\}$, $\delta(a, x)=\delta(c, y)=b$, $\delta(b, x)=\delta(c, x)=c$ and $\delta(b, y)=\delta(a, y)=a$. Moreover, let $\sum$ consist of all two-state automata. If **A** can be simulated isomorphically by a generalized α_1-product of automata from $\sum$, then, by the proof of Theorem 3, there exists a nontrivial partition of A having SP. But a short computation shows that only the two trivial partitions of A have SP.

Now define an automaton $\mathbf{B}=(X, B, \delta')$ such that $X=\{x, y\}$, $B=\{a, b, b', c\}$, $\delta'(a, x)=b$, $\delta'(b, x)=\delta'(b', x)=\delta'(c, x)=c$, $\delta'(a, y)=\delta'(b, y)=\delta'(b', y)=a$ and $\delta'(c, y)=b'$. It is obvious that the mapping τ of B onto A with $\tau(a)=a$, $\tau(b)=\tau(b')=b$ and $\tau(c)=c$ is a homomorphism of **B** onto **A**. Moreover, the partition π with two blocks $\{a, b'\}$ and $\{b, c\}$ has SP. Therefore, **B** is isomorphic to an α_0-product of two two-state automata (cf. [1], p. 184). This ends the proof of Theorem 4.

3. General products and α_i-products with $i>1$

Take a set A and a system $\pi_0, \ldots, \pi_n$ of partitions on A. We say that this system of partitions is *regular* if the following conditions are satisfied:

(i) π_0 has one block only,

(ii) π_n has one-element blocks only,

(iii) $\pi_0 \geqq \pi_1 \geqq \ldots \geqq \pi_n$.

Let π be a partition of A. For any $a \in A$, denote by $\pi(a)$ the block of π containing a. Moreover, set $M_{i,a}=\{\pi_{i+1}(b)\colon b \in A$ and $b \equiv a(\pi_i)\}$, where $a \in A$ and $i=0, \ldots, n-1$. Finally, let $\pi_i/\pi_{i+1}=\max\{|M_{i,a}|\colon a \in A\}$.

Consider an automaton $\mathbf{A}=(X, A, \delta)$. Then $(X^*)_{g(\mathbf{A})}$ always denotes a generating set of $S(\mathbf{A})$.

Now we prove.

Theorem 5. *Let $l>2$ be a natural number and $i>1$. For an automaton $\mathbf{A}=(X, A, \delta)$, $\mathbf{A}^*$ is isomorphic to some $\mathbf{B}^*$, where* **B** *is a subautomaton of a generalized*

α_i-product of automata having fewer states than l, if and only if for some $(X^*)_{g(\mathbf{A})}$ there exists a regular system $\pi_0, \dots, \pi_n$ of partitions of A such that

(I) $\pi_j/\pi_{j+1} \leqq l$ *for all* $j=0, \dots, n-1$,

(II) $a \equiv b(\pi_j)$ *implies* $\delta^*(a, x^*) \equiv \delta^*(b, x^*)$ (π_{j-i+1}) *for all* $i-1 \leqq j \leqq n$, $x^* \in (X^*)_{g(\mathbf{A})}$ *and* $a, b \in A$.

Proof. Assume that for $\mathbf{A}=(X, A, \delta)$, $\mathbf{A}^*$ is isomorphic to $\mathbf{B}^*$, where $\mathbf{B}$ is a subautomaton of a generalized α_i-product $\prod_{j=1}^{n} \mathbf{A}_j[X', \varphi]$ of automata with $|A_j| \leqq l$, $l>2$ and $i>1$. By Lemma 2, $\mathbf{B}$ is isomorphic to a subautomaton of the α_i-product $\mathbf{A}'= =(X', \bar{A}, \bar{\delta})= \prod_{j=1}^{n} \mathbf{A}_j^*[X', \varphi^*]$. We may assume that $\mathbf{B}^*$ is a subautomaton of $\mathbf{A}'^*$. Moreover, let σ: $S(\mathbf{A}) \to S(\mathbf{B})$, η: $A \to B$ be an isomorphism of $\mathbf{A}^*$ onto $\mathbf{B}^*$. Define partitions π_j $(j=1, \dots, n)$ on A in the following way: $a \equiv a'(\pi_j)$ if and only if $\eta(a)= =(a_1, \dots, a_n)$, $\eta(a')=(a_1', \dots, a_n')$ and $a_1=a_1', \dots, a_j=a_j'$. It is obvious that $\pi_0, \pi_1, \dots$ $\dots, \pi_n$ is a regular system of partitions. Moreover, condition (I) is satisfied by this system. Indeed, if $\eta(a)=(a_1, \dots, a_n)$ and $\eta(a')=(a_1', \dots, a_n')$ then $\pi_{j+1}(a') \in M_{j,a}$ if and only if $a_1'=a_1, \dots, a_j'=a_j$. Therefore, $M_{j,a}$ contains at most $|A_{j+1}|(\leqq l)$ blocks of π_{j+1}.

In order to prove the necessity of these conditions it remains to show that the system $\pi_0, \pi_1, \dots, \pi_n$ satisfies (II) as well. Denote by $(X^*)_{g(\mathbf{A})}$ the subset of $S(\mathbf{A})$ consisting of all $[p]$ $(p \in F(X))$ for which $\sigma([p])$ contains an $x' \in X'$. Since the set $\{\sigma([p]) : [p] \in (X^*)_{g(\mathbf{A})}\}$ obviously generates $S(\mathbf{B})$ thus $(X^*)_{g(\mathbf{A})}$ is a generating system of $S(\mathbf{A})$.

Take a j with $i-1 \leqq j \leqq n$, and two elements $a, a' \in A$ such that $a \equiv a'(\pi_j)$. Assume that $\eta(a)=(a_1, \dots, a_n)$ and $\eta(a')=(a_1', \dots, a_n')$. Then, by the definition of π_j, we have $a_1=a_1', \dots, a_j=a_j'$. Now choose an arbitrary $x^* \in (X^*)_{g(\mathbf{A})}$, and let $x' \in X'$ such that $x' \in \sigma(x^*)$. Moreover, let $\varphi^*(\eta(a), x')=(x_1^*, \dots, x_n^*)$ and $\varphi^*(\eta(a'), x')=(\bar{x}_1^*, \dots, \bar{x}_n^*)$. Thus, by the definition of the α_i-product, $x_1^*=\bar{x}_1^*, \dots, x_{j-i+1}^*=\bar{x}_{j-i+1}^*$ since $a_1= =a_1', \dots, a_j=a_j'$. Therefore, for $\bar{\delta}(\eta(a), x')=(b_1, \dots, b_n)$ and $\bar{\delta}(\eta(a'), x')=(b_1', \dots, b_n')$ we have $b_1=b_1', \dots, b_{j-i+1}=b_{j-i+1}'$, showing that

$$\delta^*(a, x^*) \equiv \delta^*(a', x^*)\,(\pi_{j-i+1}).$$

Conversely, assume that for an $\mathbf{A}=(X, A, \delta)$ and $(X^*)_{g(\mathbf{A})}$ there exists a regular system $\pi_0, \pi_1, \dots, \pi_n$ of partitions satisfying conditions (I) and (II). We construct automata $\mathbf{A}_j=(X_j, A_j, \delta_j)$ $(j=1, \dots, n)$ with $|A_j|=\pi_{j-1}/\pi_j(\leqq l)$ such that for a subautomaton $\mathbf{B}$ of an α_i-product of the $\mathbf{A}_j$ we have $\mathbf{A}^* \cong \mathbf{B}^*$.

Let A_j be arbitrary abstract sets with $|A_j|=\pi_{j-1}/\pi_j$. Moreover, $X_j=A_1 \times \dots$ $\dots \times A_{j+i-1} \times (X^*)_{g(\mathbf{A})}$ if $j+i-1 \leqq n$, and $X_j=A_1 \times \dots \times A_n \times (X^*)_{g(\mathbf{A})}$ otherwise.

Now let $\varkappa_j$ be a mapping of $M_j=\{\pi_j(a): a \in A\}$ onto A_j such that the restriction of $\varkappa_j$ to any $M_{j-1,a}$ is one-to-one. Define the transition function δ_j by the following rules:

(1) $j \leqq n-i+1$. Then $\delta_j(a_j, (b_1, \ldots, b_{j+i-1}, x^*)) = \varkappa_j(\pi_j(\delta^*(a, x^*)))$ $(a_j \in A_j;$ $(b_1, \ldots, b_{j+i-1}) \in A_1 \times \ldots \times A_{j+i-1}$ and $x^* \in (X^*)_{g(\mathbf{A})})$ if $a_j = b_j$ and there exists an $a \in A$ such that $\varkappa_t(\pi_t(a)) = b_t$ for all $t = 1, \ldots, j+i-1$.

(2) $j > n-i+1$. Then $\delta_j(a_j, (b_1, \ldots, b_n, x^*)) = \varkappa_j(\pi_j(\delta^*(a, x^*)))$ if $a_j = b_j$ and there exists an $a \in A$ with $\varkappa_t(\pi_t(a)) = b_t$ $(t = 1, \ldots, n)$.

(3) In all other cases δ_j is defined arbitrarily.

First we prove that δ_j is well defined. Assume that in case (1) there exists a $b \in A$ with $\varkappa_t(\pi_t(b)) = b_t$ $(t = 1, \ldots, j+i-1)$. It is enough to show that $b \equiv a$ (π_{j+i-1}) (since this, by condition (II), implies that $\delta^*(b, x^*) \equiv \delta^*(a, x^*)$ (π_j) for any $x^* \in (X^*)_{g(\mathbf{A})})$. We proceed by induction on t. $b \equiv a(\pi_1)$ obviously holds since $\varkappa_1$ is a 1—1 mapping of M_1 onto A_1. Assume that our statement has been proved for $t-1$ $(1 \leqq t-1 < j+i-1)$, i.e., $b \equiv a(\pi_{t-1})$. Therefore, since $\varkappa_t$ is 1—1 on $M_{t-1,a}$ and $\varkappa_t(\pi_t(b)) = \varkappa_t(\pi_t(a))$ thus $\pi_t(b) = \pi_t(a)$.

Case (2) can be proved by a similar argument. Note that π_n is induced by the equality relation on A. Therefore, in case (2) we get $a = b$.

Now let us form the following α_i-product $\mathbf{C} = ((X^*)_{g(\mathbf{A})}, C, \delta_{\mathbf{C}}) = \prod_{j=1}^{n} \mathbf{A}_j[(X^*)_{g(\mathbf{A})}, \varphi]$, where $\varphi = (\varphi_1, \ldots, \varphi_n)$ and for any $j = 1, \ldots, n$, $(a_1, \ldots, a_n) \in A_1 \times \ldots \times A_n$ and $x^* \in (X^*)_{g(\mathbf{A})}$,

$$\varphi_j(a_1, \ldots, a_n, x^*) = \begin{cases} (a_1, \ldots, a_{j+i-1}, x^*) & \text{if } j \leqq n-i+1, \\ (a_1, \ldots, a_n, x^*) & \text{otherwise.} \end{cases}$$

It is clear that $\mathbf{C}$ is an α_i-product.

Define a mapping $\tau \colon A \to C$ in the following way:

$$\tau(a) = (\varkappa_1(\pi_1(a)), \ldots, \varkappa_n(\pi_n(a)))$$

for any $a \in A$. We prove that τ is an isomorphism of the automaton $((X^*)_{g(\mathbf{A})}, A, \delta^*)$ into $\mathbf{C}$. First we show, by induction, that τ is 1—1. Assume that $a \neq a'$ $(a, a' \in A)$. Let t be the greatest index for which $\pi_t(a) = \pi_t(a')$. $t < n$, since otherwise $a = a'$, contradicting our assumption. Then $\pi_{t+1}(a) \neq \pi_{t+1}(a')$. Therefore, $\varkappa_{t+1}(a) \neq \varkappa_{t+1}(a')$, since $\varkappa_{t+1}$ is one-to-one on $M_{t,a}$.

Now take an arbitrary input signal $x^* \in (X^*)_{g(\mathbf{A})}$. Then

$$\delta_{\mathbf{C}}(\tau(a), x^*) = (\delta_1(\varkappa_1(\pi_1(a)), (\varkappa_1(\pi_1(a)), \ldots, \varkappa_i(\pi_i(a)), x^*)), \ldots$$

$$\ldots, \delta_n(\varkappa_n(\pi_n(a)), (\varkappa_1(\pi_1(a)), \ldots, \varkappa_n(\pi_n(a)), x^*))) =$$

$$= (\varkappa_1(\pi_1(\delta^*(a, x^*))), \ldots, \varkappa_n(\pi_n(\delta^*(a, x^*)))) = \tau(\delta^*(a, x^*)),$$

showing that τ is an isomorphism of $((X^*)_{g(\mathbf{A})}, A, \delta^*)$ onto the subautomaton $\mathbf{B} = ((X^*)_{g(\mathbf{A})}, B, \delta^*)$ of $\mathbf{C}$, where $B = \{\tau(a) | a \in A\}$. This obviously implies that τ defines an isomorphism of $\mathbf{A}^*$ onto $\mathbf{B}^*$, which completes the proof of Theorem 5.

Let us denote by $\mathbf{A}^{(2)}=(X^{(2)}, A^{(2)}, \delta^{(2)})$ the automaton for which $X^{(2)}=\{x^{(1)}, x^{(2)}\}$, $A^{(2)}=\{a^{(1)}, a^{(2)}\}$, $\delta^{(2)}(a^{(1)}, x^{(1)})=\delta^{(2)}(a^{(2)}, x^{(2)})=a^{(2)}$ and $\delta^{(2)}(a^{(2)}, x^{(1)})=\delta^{(2)}(a^{(1)}, x^{(2)})=a^{(1)}$.

Theorem 6. *Every automaton can be simulated isomorphically by a generalized α_2-power of $\mathbf{A}^{(2)}$.*

Proof. Let $\mathbf{A}=(X, A, \delta)$ be an arbitrary automaton. It is obvious that $\mathbf{T}_n=(T_n, N, \delta_n)$ with $n \geqq \max\{3, |A|\}$ isomorphically simulates $\mathbf{A}$. Therefore, in order to prove Theorem 6, by Lemma 1, it is enough to show that $\mathbf{T}_n$ can be simulated isomorphically by an α_2-power of $\mathbf{A}^{(2)}$.

Take the following elements t_1, t_2 and t_3 of T_n

$t_1(i)=i+1$ if $i<n$, and $t_1(n)=1$;

$t_2(1)=2$, $t_2(2)=1$, and $t_2(i)=i$ if $i>2$;

$t_3(1)=t_3(2)=1$, and $t_3(i)=i$ if $i>2$.

It can be proved (cf. [7]) that $\{[t_1], [t_2], [t_3]\}=(T_n^*)_{g(\mathbf{T}_n)}$ generates $S(\mathbf{T}_n)$.

First we prove that $\mathbf{T}_n$ can be simulated isomorphically by a generalized α_2-product of two-state automata. By Theorem 5, it is enough to show that there exists a regular system $\pi_0, \pi_1, \ldots, \pi_k$ of partitions of N such that

(i) $\pi_j/\pi_{j+1} \leqq 2$ for all $j=0, \ldots, k-1$;

(ii) $b \equiv c(\pi_j)$ implies that $\delta_n^*(b, t^*) \equiv \delta_n^*(c, t_n^*)(\pi_{j-1})$ for all $b, c \in N$, $t^* \in \{[t_1], [t_2], [t_3]\}$ and $1 \leqq j \leqq k$.

Let π_1 consist of the following two blocks: $\{1, \ldots, k\}$ and $\{k+1, \ldots, n\}$, where $k=u$ if $n=2u$, and $k=u+1$ if $n=2u+1$. Let us assume that the partitions π_t have been defined for all $t \leqq m \leqq k$, and that π_m has the following blocks: $\{1, \ldots, k-m+1\}$, $\{k-m+2\}, \ldots, \{k\}$, $\{k+1, \ldots, k+n-m+1\}$, $\{k+n-m+2\}, \ldots, \{n\}$. Then π_{m+1} is defined to be the partition having the blocks:

$$\{1, \ldots, k-m\}, \{k-m+1\}, \ldots, \{k\}, \{k+1, \ldots, k+n-m\}, \{k+n-m+1\}, \ldots, \{n\}.$$

It is obvious that the resulting system of partitions $\pi_0, \pi_1, \ldots, \pi_k$ is regular and satisfies (i). Moreover, (ii) obviously holds for π_1 and π_k. Now take an arbitrary m with $1 \leqq m < k-1$, and let $b, c \in N$ such that $b \equiv c(\pi_{m+1})$. We may assume that $b \neq c$. Then either $1 \leqq b, c \leqq k-m$ or $k+1 \leqq b, c \leqq k+n-m$. In the first case for any $t^* \in \{[t_1], [t_2], [t_3]\}$, $1 \leqq \delta_n^*(b, t^*)$, $\delta_n^*(c, t^*) \leqq k-m+1$, and in the second case $k+1 \leqq \delta_n^*(b, t^*)$, $\delta_n^*(c, t^*) \leqq k+n-m+1$, showing that (ii) holds for any π_j $(1 \leqq j \leqq k)$. Thus we have proved that $\mathbf{A}$ can be simulated isomorphically by a generalized α_2-product of two-state automata.

One can easily prove that every two-state automaton is isomorphic to an α_1-power of $\mathbf{A}^{(2)}$, having one factor only. Since an α_2-product of α_1-products with single factors is an α_2-product, thus $\mathbf{A}$ can be simulated isomorphically by a generalized α_2-power of $\mathbf{A}^{(2)}$.

Theorem 7. *A system $\sum$ of automata is homomorphically S-complete with respect to the generalized product if and only if there exist an* $\mathbf{A}=(X, A, \delta)\in\sum$, $a\in A$ *and* $p_1, p_2, q_1, q_2\in F(X)$ *such that* $ap_1\neq ap_2$ and $a=ap_1q_1=ap_2q_2$.

Proof. The necessity of these conditions can be proved in the same way as that of the corresponding statement for products in [9].

Conversely, assume that the conditions of Theorem 7 are satisfied by $\sum$. Set $a_1=ap_1$ and $a_2=ap_2$. Now form the following generalized α_1-product $\mathbf{B}= =(X^{(2)}, A, \delta')=(\mathbf{A})[X^{(2)}, \varphi]$, where $\varphi(a_1, x^{(1)})=q_1p_2$, $\varphi(a_1, x^{(2)})=q_1p_1$, $\varphi(a_2, x^{(1)})= =q_2p_1$ and $\varphi(a_2, x^{(2)})=q_2p_2$; moreover, $\varphi(a, x)$ is defined arbitrarily if $a\neq a_1, a_2$ $(a\in A, x\in X^{(2)})$. It is obvious that the mapping $\eta\colon a^{(j)}\to a_j$ $(j=1, 2)$ is an isomorphism of $\mathbf{A}^{(2)}$ into $\mathbf{B}$. Thus, by Theorem 6, we get that $\sum$ is *isomorphically* S*-complete* with respect to the generalized α_2-product. This ends the proof of Theorem 7.

The proof of the sufficiency of Theorem 7 yields the following

Corollary. *A system $\sum$ of automata is homomorphically S-complete with respect to the generalized product if and only if for any $i=2, 3, \ldots$, $\sum$ is isomorphically S-complete with respect to the generalized α_i-product.*

Now we are going to prove a stronger result. First we introduce the following notation, and prove a lemma.

Let us denote by $\mathbf{E}_{(2)}=(X^{(2)}, E_2, \delta^{(2)})$ the automaton for which $X^{(2)}=\{x, x_e\}$, $E_2=\{e_1, e_2\}$, $\delta^{(2)}(e_1, x_e)=e_1$, $\delta^{(2)}(e_2, x_e)=e_2$, and $\delta^{(2)}(e_i, x)=e_2$ for $i=1, 2$.

Lemma 3. *Let $\mathbf{B}=(Y, B, \delta)$ be an automaton such that there exists a well ordering $\leqq$ on B with the property that $b\leqq bp$ for any $b\in B$ and $p\in F(Y)$. Then B is isomorphic to a subautomaton of an α_0-power of* $\mathbf{E}_{(2)}$.

Proof. Assume that the conditions of Lemma 3 are satisfied. Moreover, let $B=\{b_1, \ldots, b_n\}$, and $b_i<b_j$ if $i<j$. Now define partitions π_t $(t=1, \ldots, n-1)$ on B in the following way: $b_u\equiv b_v(\pi_t)$ implies $b_u=b_v$ if $u\leqq t$ or $v\leqq t$, and $b_u\equiv b_v$ (π_t) for all $u, v>t$. It is obvious that all π_t have SP, $\pi_1>\pi_2>\ldots>\pi_{n-1}$ and $\pi_t/\pi_{t+1}=2$.

For any $t(=1, \ldots, n-1)$ take an abstract set $A_t=\{a_t^{(1)}, a_t^{(2)}\}$. Furthermore, define mappings $\varkappa_t$ of $M_t=\{\pi_t(b)|b\in B\}$ onto A_t such that $\varkappa_t(\{b_j\})=a_t^{(1)}$ if $j\leqq t$ and $\varkappa_t(\{b_{t+1}, \ldots, b_n\})=a_t^{(2)}$. Obviously, $\varkappa_t$ is 1—1 on $M_{t-1,b}$ for any $b\in B$. (π_0 is the trivial partition of B having one block only.)

Now let us define the automata $\mathbf{A}_t=(X_t, A_t, \delta_t)$ in the following way: $X_1=Y$, and $X_t=A_1\times\ldots\times A_{t-1}\times Y$ if $1<t<n$. Moreover, $\delta_1(a_1, y)=\varkappa_1(\pi_1(\delta(b, y)))$ $(a_1\in A_1, y\in Y)$, where $b\in\varkappa^{-1}(a_1)$, and

(i) $\delta_t(a_t, (a_1, \ldots, a_{t-1}, y))=\varkappa_t(\pi_t(\delta(b, y)))$ if there exists a $b\in B$ such that $\varkappa_j(\pi_j(b))=a_j$ $(j=1, \ldots, t)$;

(ii) $\delta_t(a_t, (a_1, \ldots, a_{t-1}, y))=a_t$ otherwise,

where $y\in Y$ and $(a_1, \ldots, a_t)\in A_1\times\ldots\times A_t$.

Now form the α_0-product $\mathbf{C}=(Y, C, \delta_{\mathbf{C}})=\prod_{t=1}^{n-1} \mathbf{A}_t[Y, \varphi]$ for which $\varphi_1(a_1, \ldots$ $\ldots, a_{n-1}, y)=y$, and $\varphi_t(a_1, \ldots, a_{n-1}, y)=(a_1, \ldots, a_{t-1}, y)$ if $t>1$ $(y\in Y, a_j\in A_j,$ $j=1, \ldots, n-1)$. One can prove in a way similar to that in the proof of the sufficiency of Theorem 5, that the mapping $\tau: b\to(\varkappa_1(\pi_1(b)),\ldots, \varkappa_{n-1}(\pi_{n-1}(b)))$ is an isomorphism of $\mathbf{B}$ into $\mathbf{C}$.

Now let us order the elements of A_t by $a_t^{(1)}<a_t^{(2)}$. We prove that for any $x_t\in X_t$, $\delta_t(a_t^{(i)}, x_t)=a_t^{(j)}$ $(1\leqq i, j\leqq 2)$ implies $a_t^{(i)}\leqq a_t^{(j)}$. Take an arbitrary $x_t=(a_1, \ldots, a_{t-1}, y)\in X_t$. If there exists no $b\in B$ with $\varkappa_s(\pi_s(b))=a_s$ $(s=1, \ldots, t-1)$ and $\varkappa_t(\pi_t(b))=a_t^{(i)}$ then, by (ii) in Lemma 3, $\delta_t(a_t^{(i)}, x_t)=a_t^{(i)}$. Now assume that for a $b_u\in B$, $\varkappa_s(\pi_s(b_u))=a_s$ $(s=1, \ldots, t;\ a_t=a_t^{(i)})$ and $\delta(b_u, y)=b_v$. Then $b_u\leqq b_v$. Therefore, by the definition of $\varkappa_t$ and the ordering on A_t, $\varkappa_t(\pi_t(b_u))=a_t^{(i)}\leqq a_t^{(j)}=\varkappa_t(\pi_t(b_v))$.

Finally, we show that $\mathbf{A}_t$ can be represented isomorphically by an α_0-power of $\mathbf{E}_{(2)}$ (having a single factor). Take the α_0-power $\mathbf{D}_t=(X_t, E_2, \delta_{\mathbf{D}})=(\mathbf{E}_{(2)})[X_t, \psi]$, where for any $e_i\in E_2$ and $x_t\in X_t$,

$$\psi(e_i, x_t)=\begin{cases} x & \text{if } \delta_t(a_t^{(1)}, x_t)=a_t^{(2)}, \\ x_e & \text{if } \delta_t(a_t^{(1)}, x_t)=a_t^{(1)}. \end{cases}$$

It can be shown, by a short computation, that the mapping $\eta: a_t^{(i)}\to e_i$ $(i=1, 2)$ is an isomorphism of $\mathbf{A}_t$ onto $\mathbf{D}_t$.

Since the formation of the α_0-product is associative, thus we proved that $\mathbf{B}$ can be represented isomorphically by an α_0-power of $\mathbf{E}_{(2)}$.

Now we prove

Theorem 8. *Let $\sum$ be an arbitrary set of automata. An automaton $\mathbf{B}$ can be simulated homomorphically by a generalized product of automata from $\sum$ if and only if $\mathbf{B}$ can be simulated isomorphically by a generalized α_2-product of automata from $\sum$.*

Proof. If there is an $\mathbf{A}\in\sum$ satisfying the conditions of Theorem 7 then, by the Corollary to Theorem 7, $\sum$ is isomorphically S-complete with respect to the generalized α_2-product. Therefore, in the sequel we may assume that none of the automata in $\sum$ satisfies the conditions of Theorem 7.

Let $\mathbf{B}=(Y, B, \delta)$ be an automaton which can be simulated homomorphically by a generalized product of automata from $\sum$. It can be shown that $\mathbf{B}$ does not satisfy the conditions of Theorem 7. Consequently, one can define a well ordering $\leqq$ on B such that for any $b, c\in B$ and $p\in F(Y)$, $bp=c$ implies $b\leqq c$. Now assume that there exist $b, c\in B$ and $p\in F(Y)$ with $bp=c$ and $b\neq c$. It is easy to prove that in this case there exist an $\mathbf{A}=(X, A, \delta')$ in $\sum$, $a_1, a_2\in A$, $p_1, p_2\in F(Y)$ such that $a_1p_1=a_2p_1=$ $=a_2p_2=a_2$, $a_1p_2=a_1$ and $a_1\neq a_2$.

By Lemma 3, $\mathbf{B}$ can be represented isomorphically by an α_0-power of $\mathbf{E}_{(2)}$. Since the formation of the generalized α_0-product is associative, thus it is enough to show that $\mathbf{E}_{(2)}$ can be represented isomorphically by a generalized α_0-power of $\mathbf{A}$. Take the α_0-power $\mathbf{D}=(X^{(2)}, A, \delta_{\mathbf{D}})=(A^*)[X^{(2)}, \psi]$, where for any $a\in A$, $\psi(a, x)=[p_1]$ and $\psi(a, x_e)=[p_2]$. Then $\tau: e_i\to a_i$ $(i=1, 2)$ defines an isomorphism of $\mathbf{E}_{(2)}$ into $\mathbf{D}$.

Now if for any $b\in B$ and $y\in Y$, $\delta(b, y)=b$ and B has at least two elements then there exists an $\mathbf{A}\in\sum$ such that $\mathbf{A}$ has at least two states. Then $\mathbf{B}$ can be represented isomorphically by a generalized α_0-power of $\mathbf{A}$. Finally, if $|B|=1$ then $\mathbf{B}$ can be represented isomorphically by a generalized α_0-power of any automata from $\sum$. This ends the proof of Theorem 8.

4. T-products and (T, α_i)-products $(i=0,1,\dots)$

In [8] G. I. Ivanov introduced the concept of the temporal composition as an abstract equivalent of the single-channel representation of multichannel finite state machines (see [5]). Now we restrict the definition of the temporal composition to automata.

Let $\mathbf{A}_i=(X_i, A, \delta_i)$ $(i=1,2)$ be arbitrary automata having a common state set A. Take a set X with $|X|=|X_1\times X_2|$ and a 1—1 mapping γ of X onto $X_1\times X_2$. Then the automaton $\mathbf{A}=(X, A, \delta)$ is the *temporal product* of $\mathbf{A}_1$ by $\mathbf{A}_2$ with respect to X and γ if for any $a\in A$ and $x\in X$, $\delta(a, x)=\delta_2(\delta_1(a, x_1), x_2)$, where $(x_1, x_2)=\gamma(x)$.

The concept of the temporal product can be generalized in a natural way for arbitrary finite family of automata. It should be noted that the formation of the temporal product is associative.

We say that an automaton $\mathbf{A}$ is a *(T, α_i)-product* $(i=0, 1, \dots)$ [*T-product*] of automata from $\sum$ if there exists a sequence of classes of automata, $\sum=\sum_0, \sum_1, \sum_2, \sum_3$ such that the automata in $\sum_1$ and $\sum_3$ can be given as temporal products of automata in $\sum_0$ and $\sum_2$, respectively, the automata in $\sum_2$ are isomorphic copies of subautomata of α_i-products [products] of automata from $\sum_1$, and $A\in\sum_3$.

Let us note that in the definition of $\sum_2$ it would be enough to confine ourselves to isomorphic copies of α_i-products [products] of automata in $\sum_1$. However, it would make our computations more difficult, without yielding any further results.

In the sequel we assume that if $\sum$ is a system of automata then for any $\mathbf{A}=(X, A, \delta)\in\sum$ there exists an $x\in X$ inducing the identity mapping of A, i.e., $\delta(a, x)=a$ for all $a\in A$.

We say that an automaton $\mathbf{A}$ *can be represented homomorphically by a T-product* [*(T, α_i)-product*] of automata from $\sum$ if $\mathbf{A}$ is a homomorphic image of a subautomaton of a T-product [(T, α_i)-product] of automata in $\sum$. The concept of the *isomorphic representation* is defined similarly. Moreover, $\sum$ is *homomorphically complete* with respect to the T-product [(T, α_i)-product] if every automaton can be represented homomorphically by a T-product [(T, α_i)-product] of automata from $\sum$. A natural modification of this definition leads to the concept of the *isomorphic completeness* with respect to the T-product [(T, α_i)-product].

The following results show the relation between simulations by generalized products and representations by T-products and (T, α_i)-products of automata. One can easily prove that if $\sum$ is a system of automata and $\mathbf{A} \in \sum$ then $\mathbf{A}^*$ can be represented isomorphically by a temporal power of $\mathbf{A}$. Thus we have

Theorem 9. *If $\sum$ is isomorphically (homomorphically) S-complete with respect to the generalized α_0-product then $\sum$ is isomorphically (homomorphically) complete with respect to the (T, α_0)-product.*

The converse of Theorem 9 fails to hold which will follow from Theorems 1 and 11.

Theorem 10. *Assume that a set $\sum$ of automata is homomorphically complete with respect to the (T, α_0)-product. Then there exist an $\mathbf{A}=(X, A, \delta) \in \sum$, $a, b \in A$ and a word $p \in F(X)$ such that $a \neq b$ and $ap=bp=b$.*

Proof. Let $\sum_1$, $\sum_2$ and $\sum_3$ denote the same classes of automata as in the definition of the (T, α_0)-product.

Assume that $\sum$ is homomorphically complete with respect to the (T, α_0)-product. Then there exists a $\mathbf{B}=(X, B, \delta)$ in $\sum_3$ such that $\mathbf{E}_{(2)}$ is a homomorphic image of a subautomaton of $\mathbf{B}$. (For the definition of $\mathbf{E}_{(2)}$, see p. 32.) One can prove that there exist $a, b \in B$, $x \in X$ and a positive integer k such that $a \neq b$ and $ap=bp=b$, where $p=x^k$.

Suppose that $\mathbf{B}$ is a temporal product of $\mathbf{B}_1, \ldots, \mathbf{B}_l$ with respect to X and γ such that $\mathbf{B}_i=(X_i, B, \delta_i)$ $(i=1, \ldots, l)$, $\mathbf{B}_i \in \sum_2$ and $\gamma(x)=(x_1, \ldots, x_l)(\in X_1 \times \ldots \times X_l)$. For any $t(=0, 1, \ldots)$ and $1 \leqq i < l$, let $a_{t \cdot l+i}$ and $b_{t \cdot l+i}$ denote the elements $a(x^t)_{\mathbf{B}}(x_1)_{\mathbf{B}_1} \ldots (x_i)_{\mathbf{B}_i}$ and $b(x^t)_{\mathbf{B}}(x_1)_{\mathbf{B}_1} \ldots (x_i)_{\mathbf{B}_i}$, respectively. Thus, $a=a_0$, $b=b_0=a_{k \cdot l}=b_{k \cdot l}$. Now assume that $u<k \cdot l$ is the greatest nonnegative integer for which $a_u \neq b_u$. There exists such a u, since $a_0 \neq b_0$. Let u be given in the form $u=m \cdot l+v$, where m and v are nonnegative integers and $v<l$. Therefore, $\delta_{v+1}(a_u, x_{v+1})=\delta_{v+1}(b_u, x_{v+1})$. This means that there are $c, d \in B$ and a positive integer n such that $c \neq d$ and $c(x_{v+1}^n)_{\mathbf{B}_{v+1}}=d(x_{v+1}^n)_{\mathbf{B}_{v+1}}=d$.

Thus we have got that there exist a $\mathbf{C}=(Y, C, \delta_C)$ in $\sum_2$, $c, d \in C$, $y \in Y$ and a positive integer k such that $c \neq d$ and $cy^k=dy^k=d$. Assume that $\mathbf{C}$ can be given by an α_0-product $\mathbf{C}=(\mathbf{C}_1 \times \mathbf{C}_2)[Y, \varphi]$, where $\mathbf{C}_i=(Y_i, C_i, \delta_i')$ $(i=1, 2)$. Let $c=(c_1, c_2)$ and $d=(d_1, d_2)$. For a $p=y_1 \ldots y_n \in F(Y)$ and $c' \in C_1$ let $p(\mathbf{C}_1)=\varphi_1(y_1) \ldots \varphi_1(y_n)$ and $p(\mathbf{C}_2, c')=y_1' \ldots y_n'$, where $y_1'=\varphi_2(c', y_1), \ldots, y_n'=\varphi_2(c'(y_1 \ldots y_{n-1})(\mathbf{C}_1), y_n)$. Then, for $q=y^k$, we obviously have $c_1 q(\mathbf{C}_1)=d_1$, $d_1 q(\mathbf{C}_1)=d_1$ and $c_2 q(\mathbf{C}_2, c_1)=d_2$, $d_2 q(\mathbf{C}_2, d_1)=d_2$. Now if $c_1 \neq d_1$ then there exists a word $q'=q(\mathbf{C}_1) \in F(Y_1)$ such that $c_1 q'=d_1 q'=d_1$.

Let us assume that $c_1=d_1$. Then $q(\mathbf{C}_2, c_1)=q(\mathbf{C}_2, d_1)$, and $c_2\neq d_2$ since $c\neq d$. Therefore, in this case for $q''=q(\mathbf{C}_2, c_1)\in F(Y_2)$ we have $c_2q''=d_2q''=d_2$.

Since $\mathbf{C}\in\sum_2$ and the formation of the α_0-product is associative, thus we have got that there exist an automaton $\mathbf{D}=(Z, D, \delta_{\mathbf{D}})$ in $\sum_1$, two states $d, d'\in D$ and a word $p\in F(Z)$ such that $d\neq d'$ and $dp=d'p=d'$. Assume that $p=z_1...z_n$ $(z_i\in Z)$. Let us denote by d_i and d_i' the states dp_i and $d'p_i$, respectively, where p_i is the prefix of p of length i, for all $0\leqq i<n$. Suppose that $j<n$ is the greatest nonnegative integer with $d_j\neq d_j'$. Since $d_0\neq d_0'$ thus there exists such a j. Therefore, $\delta_{\mathbf{D}}(d_j, z_{j+1})=\delta_{\mathbf{D}}(d_j', z_{j+1})$. Thus, there are states $a', b'\in D$ and a positive integer t such that $a'\neq b'$ and $a'z_{j+1}^t=b'z_{j+1}^t=b'$. Now, since $\mathbf{D}$ is a temporal product of automata from $\sum$ thus there exist an $\mathbf{A}=(X, A, \delta)\in\sum$, $a, b\in A$ and a word $p\in F(X)$ such that $a\neq b$ and $ap=bp=b$. (See the proof of the similar statement concerning $\mathbf{B}$.) This ends the proof of Theorem 10.

Take an automaton $\mathbf{A}=(X, A, \delta)$, a state $a\in A$ and an input signal $x\in X$. Then the *cycle* generated by (a, x) in $\mathbf{A}$ means the set of elements $ax^0, ax, \ldots, ax^k, \ldots$. For this cycle we use the short notation (a, x). If $ax^0, \ldots, ax^u$ are pairwise different and u is the least exponent for which there exists a $w>u$ such that $ax^w=ax^u$ then $ax^0, \ldots, ax^{u-1}$ is the *preperiod* of (a, x) and u is the *length of* this *preperiod.* (When the preperiod is empty its length equals 0.) Furthermore, if $u+v$ is the smallest positive integer for which $ax^u=ax^{u+v}$ holds then $ax^u, \ldots, ax^{u+v-1}$ is the *period* of the cycle under question, and v is the *length of* this *period.* In this case we say that (a, x) is a cycle of *type* (u, v).

An automaton $\mathbf{A}=(X, A, \delta)$ is called *x-cyclic* $(x\in X)$ of *type* (k, l) if for some $a\in A$, the set A coincides with the cycle (a, x) in $\mathbf{A}$, and this cycle is of type (k, l), while the input signals different from x induce the identity mapping of A. $\mathbf{A}$ is said to be a *prime-power automaton* with respect to x if it is x-cyclic of type $(0, r^n)$, where r is a prime and n is a natural number. If $n=1$ then $\mathbf{A}$ is a *prime automaton.* Moreover, $\mathbf{A}$ is an *elevator* regarding x if it is x-cyclic of type $(k, 1)$ with $k\geqq 1$.

For any natural number r, let $\mathbf{C}_{(r)}=(X, C_r, \delta_r)$ denote the following automaton: $X=\{x, x_e\}$, $C_r=\{c_0^{(r)}, \ldots, c_{r-1}^{(r)}\}$, $\delta_r(c_j^{(r)}, x_e)=c_j^{(r)}$ $(0\leqq j<r)$ and $\delta_r(c_j^{(r)}, x)=c_{(j+1)(\mathrm{mod}\, r)}^{(r)}$. Moreover, let $\mathbf{E}_{(t)}=(X, E_t, \delta^{(t)})$ be the elevator of type $(t, 1)$, where $X=\{x, x_e\}$, $E_t=\{e_1, \ldots, e_t\}$, $\delta^{(t)}(e_j, x_e)=e_j$ $(j=1, \ldots, t)$, $\delta^{(t)}(e_j, x)=e_{j+1}$ if $j<t$, and $\delta^{(t)}(e_t, x)=e_t$. Finally, let $\sum_P$ denote the system consisting of $\mathbf{E}_{(2)}$ and of $\mathbf{C}_{(r)}$ for all prime number r.

Now we prove

Lemma 4. *Let* $\mathbf{A}=(X, A, \delta)$ *be an automaton with two input signals such that one of them induces the identity mapping of* A. *Then* $\mathbf{A}$ *can be represented isomorphically by an* α_0*-product of automata from* $\sum_P$.

Proof. Let $\mathbf{A}=(X, A, \delta)$ be an arbitrary automaton with $X=\{x, x_e\}$ such that x_e induces the identity mapping of $\mathbf{A}$. Then $\mathbf{A}$ can be given as a union of pairwise disjoint subsets $A_1, \ldots, A_k$ such that $\mathbf{A}_i=(X, A_i, \delta_i)$ $(i=\mathrm{i}, \ldots, k)$ are connected subautomata of $\mathbf{A}$, where δ_i denotes the restriction of δ to A_i.

For an $a \in A$ we say that it is *initial* if (a, x) is of type (s, r) with $s>0$ and there exists no $b \in A$ and $p \in F(X)$ such that $b \neq a$ and $bp=a$. Assume that $\{a_{i1}, \ldots, a_{il_i}\}$ is the set of all the initial elements of A_i $(i=1, \ldots, k)$. For any a_{ij} take the cycle (a_{ij}, x) in $\mathbf{A}_i$. It is obvious that these cycles (a_{ij}, x) $(j=1, \ldots, l_i)$ have the same period, say of type $(0, t_i)$. Define a partition π_{i0} on A in the following manner:

(i) for $a, b \in A_i$, $a \equiv b$ (π_{i0}) if and only if there exists a $p \in F(X)$ with $|p|=u \cdot t_i$ such that $ap=bp$,

(ii) if $a, b \notin A_i$ then $a \equiv b$ (π_{i0}),

(iii) $a \equiv b$ (π_{i0}) implies $a, b \in A_i$ or $a, b \notin A_i$. One can show, by a short computation, that π_{i0} has SP.

Now for any initial state a_{ij}, let π_{ij} be the following partition of A: the elements in the preperiod of (a_{ij}, x) as well as the elements in all preperiods having common elements with the preperiod of (a_{ij}, x) form one-element blocks of π_{ij}, and all other elements of A are in the same block of π_{ij}. Again, a short computation shows that π_{ij} has SP. Moreover, the intersection $\bigcap(\pi_{ij} | i=1, \ldots, k;\, j=0, \ldots, l_i)$ is the trivial partition having one-element blocks only. Therefore, $\mathbf{A}$ can be given as a subdirect product of the quotient automata $\mathbf{A}/\pi_{ij}$ $(i=1, \ldots, k;\, j=0, \ldots, l_i)$.

Let us consider a quotient automaton $\mathbf{A}/\pi_{ij}$ with $j>0$. Then $\mathbf{A}/\pi_{ij}$ is either a one-state automaton or it satisfies Lemma 3. If $\mathbf{A}/\pi_{ij}$ has only one state then it can be represented isomorphically by an α_0-power (having a single factor) of any automaton in $\sum_P$. In the other case, by Lemma 3, $\mathbf{A}/\pi_{ij}$ can be represented isomorphically by an α_0-power of $\mathbf{E}_{(2)}$.

Now let us investigate the quotient automaton $\mathbf{A}/\pi_{i0}$. Obviously, $(\pi_{i0}(a_{ij}), x)$ forms a cycle in $\mathbf{A}/\pi_{i0}$ of type $(0, t_i)$. (Note that this cycle is independent of j.) We distinguish the following three cases:

(1) $t_i=k=1$. Then $\mathbf{A}/\pi_{i0}$ is a one-state automaton. Therefore, it can be represented isomorphically by an α_0-power of any automaton from $\sum_P$.

(2) $t_i>1$ and $k=1$. In this case $\mathbf{A}/\pi_{i0}$ is isomorphic to $\mathbf{C}_{(t)}$. Let t_i be given in the form $t_i=r_1^{w_1} \ldots r_n^{w_n}$, where r_j are pairwise different prime numbers and $w_j>0$ $(j=1, \ldots, n)$. Then $\mathbf{C}_{(t)}$ is isomorphic to the direct product of $\mathbf{C}_{(s_1)}, \ldots, \mathbf{C}_{(s_n)}$, where $s_j=r_j^{w_j}$ (see the proof of Theorem 1 in [4]).

Take $\mathbf{C}_{(s)}$ such that $s=r^l$, where r is a prime number and $l>0$. We prove that $\mathbf{C}_{(s)}$ can be represented isomorphically by an α_0-power of $\mathbf{C}_{(r)}$. Obviously, it is enough to show that whenever $l>1$ then there exists an α_0-product of $\mathbf{C}_{(r^{l-1})}$ and $\mathbf{C}_{(r)}$ which is isomorphic to $\mathbf{C}_{(r^l)}$. Form the α_0-product $\mathbf{C}=(\mathbf{C}_{(r^{l-1})} \times \mathbf{C}_{(r)})[X, \varphi]$, where for any $y\in X$ and $(c_u^{(r^{l-1})}, c_v^{(r)})$ from $C_{r^{l-1}}\times C_r$, $\varphi_1(c_u^{(r^{l-1})}, c_v^{(r)}, y)=y$ and

$$\varphi_2(c_u^{(r^{l-1})}, c_v^{(r)}, y) = \begin{cases} x & \text{if} \quad u = r^{l-1}-1 \quad \text{and} \quad y = x, \\ x_e & \text{otherwise.} \end{cases}$$

By the definition of φ, $(c_0^{(r^{l-1})}, c_0^{(r)})x^z=(c_z^{(r^{l-1})}, c_0^{(r)})$ if $z<r^{l-1}$, and

$$(c_0^{(r^{l-1})}, c_0^{(r)})x^z = (c_0^{(r^{l-1})}, c_1) \quad \text{if} \quad z = r^{l-1}.$$

From this it can be seen immediately, that $(c_0^{(r^{l-1})}, c_0^{(r)})x^z\neq(c_0^{(r^{l-1})}, c_0^{(r)})$ if $z<r^l$, and $(c_0^{(r^{l-1})}, c_0^{(r)})x^z=(c_0^{(r^{l-1})}, c_0^{(r)})$ provided that $z=r^l$. Moreover, x_e induces the identity mapping of the state set of $\mathbf{C}$. Therefore, $\mathbf{C}$ is x-cyclic of type $(0, r^l)$, showing that $\mathbf{C}$ is isomorphic to $\mathbf{C}_{(s)}$. Since the formation of the α_0-product is associative, thus we got that $\mathbf{A}/\pi_{i0}$ can be represented isomorphically by an α_0-product of automata from Σ_p.

(3) $k>1$. Now if $t_i=1$ then $\mathbf{A}/\pi_{i0}$ has two states and both input signals induce the identity mapping of its state set. Therefore, $\mathbf{A}/\pi_{i0}$ can be represented isomorphically by an α_0-power (with a single factor) of arbitrary automata from $\sum_P$. Thus, we may assume that $t_i>1$ too. Then $\mathbf{A}/\pi_{i0}$ is isomorphic to the following automaton $\mathbf{C}=(X, C, \delta_{\mathbf{C}})$: $C=\{c, c_0, \dots, c_{t_i-1}\}$, $\delta_{\mathbf{C}}(c, x)=\delta_{\mathbf{C}}(c, x_e)=c$, $\delta_{\mathbf{C}}(c_j, x)=c_{(j+1)(\mathrm{mod}\, t_i)}$ and $\delta_{\mathbf{C}}(c_j, x_e)=c_j$ $(0\leqq j<t_i)$. We now prove that $\mathbf{C}$ can be represented isomorphically by an α_0-product of $\mathbf{E}_{(2)}$ and $\mathbf{C}_{(t_i)}$. Take $\mathbf{D}=(X, D, \delta_D)=\mathbf{E}_{(2)}\times\mathbf{C}_{(t_i)})$ $[X, \varphi]$, where for any $(e_u, c_v^{(t_i)})\in D$ and $y\in X$, $\varphi_1(e_u, c_v^{(t_i)}, y)=x_e$ and

$$\varphi_2(e_u, c_v^{(t_i)}, y) = \begin{cases} y & \text{if} \quad u = 2, \\ x_e & \text{if} \quad u = 1. \end{cases}$$

Then the mapping $\eta: C\to D$ with $\eta(c)=(e_1, c_0^{(t_i)})$ and $\eta(c_j)=(e_2, c_j^{(t_i)})$ $(0\leqq j<t_i)$ is an isomorphism of $\mathbf{C}$ into $\mathbf{D}$. Moreover, by the proof of (2), $\mathbf{C}_{(t_i)}$ can be represented isomorphically by an α_0-product of automata from $\sum_P$. Thus, we got that $\mathbf{A}/\pi_{i0}$ can be represented isomorphically by an α_0-product of automata in $\sum_P$. This completes the proof of Lemma 4.

Now we are ready to prove

Theorem 11. *A system $\sum$ of automata is isomorphically complete with respect to the (T, α_0)-product if and only if there exist an $\mathbf{A}=(X', A, \delta)\in\sum$, $a, b\in A$ and a word $p\in F(X')$ such that $a\neq b$ and $ap=bp=b$.*

Proof. The necessity of these conditions follows from Theorem 10.

Conversely, assume that in $\sum$ there is an automaton satisfying the above conditions. Again, let $\sum_1$, $\sum_2$ and $\sum_3$ denote those classes of automata as in the definition of the (T, α_0)-product.

Now take an automaton $\mathbf{C}=(Z, C, \delta_{\mathbf{C}})$ such that $Z=\{z, z_e\}$ and for any $c\in C$, $\delta_{\mathbf{C}}(c, z_e)=c$. By Lemma 4, $\mathbf{C}$ can be represented isomorphically by an α_0-product

$\mathbf{D}=(Z, D, \delta_{\mathbf{D}})=\prod_{i=1}^{n} \mathbf{B}_i[Z, \varphi]$ of automata from $\sum_P$. For any $i \leqq n$, define two automata in the following way:

(i) Assume that $\mathbf{B}_i$ is a prime automaton $\mathbf{C}_{(r)}$. Then let

$$\mathbf{C}'_{(r)} = (X, C'_r, \delta'_r), \quad \text{where} \quad X = \{x, x_e\},$$

$$C'_r = \{c_0^{(r)\prime}, c_0^{(r)*}, \ldots, c_{r-1}^{(r)\prime}, c_{r-1}^{(r)*}\},$$

$$\delta'_r(c_i^{(r)\prime}, x_e) = c_i^{(r)\prime},$$

$$\delta'_r(c_i^{(r)*}, x) = \delta'_r(c_i^{(r)*}, x_e) = c_i^{(r)*} \quad \text{and} \quad \delta'_r(c_i^{(r)\prime}, x) = c_i^{(r)*} \qquad (0 \leqq i < r).$$

Moreover, let $\mathbf{C}''_{(r)}=(X, C'_r, \delta''_r)$ be the automaton for which

$$\delta''_r(c_i^{(r)\prime}, x_e) = \delta''_r(c_i^{(r)\prime}, x) = c_i^{(r)\prime},\ \delta''_r(c_i^{(r)*}, x_e)=c_i^{(r)*}, \text{ and } \delta''_r(c_i^{(r)*}, x)=c^{(r)\prime}_{(i+1)\,(\mathrm{mod}\, r)}.$$

(ii) If $\mathbf{B}_i$ is the elevator $\mathbf{E}_{(2)}$ then we define the following two automata: $\mathbf{E}'_2=(X, E'_2, \delta'_{(2)})$ and $\mathbf{E}''_2=(X, E'_2, \delta''_{(2)})$, where $X=\{x, x_e\}$, $E'_2=\{e'_1, e^*_1, e'_2\}$ and

$\delta'_{(2)}$	x	x_e
e'_1	e^*_1	e'_1
e^*_1	e^*_1	e^*_1
e'_2	e'_2	e'_2

$\delta''_{(2)}$	x	x_e
e'_1	e'_1	e'_1
e^*_1	e'_2	e^*_1
e'_2	e'_2	e'_2

Let us form the α_0-products

$$\mathbf{D}' = (Z, D', \delta'_{\mathbf{D}}) = \prod_{i=1}^{n} \mathbf{B}'_i[Z, \varphi'] \quad \text{and} \quad \mathbf{D}'' = (Z, D', \delta''_{\mathbf{D}}) = \prod_{i=1}^{n} \mathbf{B}''_i[Z, \varphi'']$$

such that for any $(b_1, \ldots, b_n) \in D$ and $z' \in Z$,

$$\varphi'(d_1, \ldots, d_n, z') = \varphi''(d_1, \ldots, d_n, z') = \varphi(b_1, \ldots, b_n, z'),$$

where $d_i=b'_i$ or b^*_i $(i=1, \ldots, n)$. Moreover, take the temporal product $\mathbf{G}=(Z\times Z, G, \delta_{\mathbf{G}})$ of $\mathbf{D}'$ by $\mathbf{D}''$ with respect to the identity mapping γ' on $Z\times Z$. One can show that the mappings $\varkappa'\colon Z\to Z\times Z$ and $\eta\colon D\to D'$ with $\varkappa'(z')=(z', z')$ and $\eta((b_1, \ldots, b_n))=(b'_1, \ldots, b'_n)$ $(z' \in Z, (b_1, \ldots, b_n)\in D)$ is an isomorphism of $\mathbf{D}$ into $\mathbf{G}$.

It is obvious that $\mathbf{E}_{(2)}$ can be represented isomorphically by a temporal power of the automaton $\mathbf{A}$ satisfying the conditions of Theorem 11. Moreover, the well ordering $c_0^{(r)\prime}<c_0^{(r)*}<\ldots<c_{r-1}^{(r)\prime}<c_{r-1}^{(r)*}$ of the state set of $\mathbf{C}'_{(r)}$, and the well ordering $c_0^{(r)*}<c_1^{(r)\prime}<\ldots<c_{r-1}^{(r)*}<c_0^{(r)\prime}$ of the state set of $\mathbf{C}''_{(r)}$ satisfy the conditions of Lemma 4. Therefore, $\mathbf{C}'_{(r)}$ and $\mathbf{C}''_{(r)}$ can be represented isomorphically by an α_0-power of $\mathbf{E}_{(2)}$. Similarly, the well ordering $e'_1<e^*_1<e'_2$ of the state sets of $\mathbf{E}'_2$ and $\mathbf{E}''_2$ show that $\mathbf{E}'_2$ and $\mathbf{E}''_2$ can be represented isomorphically by α_0-powers of $\mathbf{E}_{(2)}$. Since the formation of the α_0-product is associative, thus we got that $\mathbf{D}', \mathbf{D}''\in \sum_2$.

Now let $\mathbf{B}=(Y, B, \delta')$ be an arbitrary automaton, and for every $y \in Y$ take $Z_y=\{y, y_e\}$ and denote by $\mathbf{B}_y=(Z_y, B, \delta_y)$ the automaton whose transition function is defined by $\delta_y(b, y)=\delta'(b, y)$ and $\delta_y(b, y_e)=b$ for any $b \in B$.

For all $\mathbf{B}_y$ take an α_0-product $\mathbf{D}_y=(Z_y, D_y, \bar{\delta}_y)=\prod_{i=1}^{n_y} \mathbf{B}_i^{(y)}[Z_y, \varphi_y]$ of prime automata $\mathbf{C}_{(r)}$ and $\mathbf{E}_{(2)}$ such that $\psi_y: B \to D_y$ is an isomorphism of $\mathbf{B}_y$ into $\mathbf{D}_y$. Without loss of generality we may assume that $D_y=D_{y'}(=D)$ and $\psi_y(b)=\psi_{y'}(b)(=\psi(b))$ for any $y, y' \in Y$ and $b \in B$. Indeed, if $\mathbf{C}_{(r)}$ is a factor in some $\mathbf{D}_{y'}$ with multiplicity m' and m_r is the maximal number of occurrences of $\mathbf{C}_{(r)}$ in the α_0-products $\mathbf{D}_{y'}$ then $\mathbf{D}_{y'}$ can be replaced by a suitable α_0-product of $\mathbf{D}_{y'}$ by $\mathbf{C}_{(r)}^{m_r-m'}$. Similar statement is valid for $\mathbf{E}_{(2)}$. (Observe that x_e always induces the identity mappings of the state sets of $\mathbf{C}_{(r)}$ and $\mathbf{E}_{(2)}$.) The requirement $\psi_y(b)=\psi_{y'}(b)$ can be satisfied by a suitable renaming of the elements of the D_y.

Now for all $y \in Y$ construct the α_0-products $\mathbf{D}'_y=(Z_y, D'_y, \delta'_y)$ and $\mathbf{D}''=(Z_y, D'_y, \delta''_y)$ (as for $\mathbf{D}$ at the beginning of the proof). It is obvious, by the construction of $\mathbf{D}'_y$ and $\mathbf{D}''_y$, that $|D'_y|=|D'_{y'}|$ for any $y, y' \in Y$. Moreover, these automata $\mathbf{D}'_y$ and $\mathbf{D}''_y$ are in $\sum_2$, and $\mathbf{D}_y$ is isomorphic to a subautomaton of the temporal product $\mathbf{G}_y$ of $\mathbf{D}'_y$ by $\mathbf{D}''_y$, under some mappings $\varkappa_y: Z_y \to Z_y \times Z_y$ and $\eta_y: D_y \to D'_y$. Again, by a suitable renaming of the elements of D'_y, we can achive that $D'_y=D'_{y'}$ $(=D')$ and $\eta_y(d)=\eta_{y'}(d)(=\eta(d))$ for all $y, y' \in Y$ and $d \in D_y$.

Assume that $Y=\{y_1, \dots, y_k\}$. Take the temporal product $\mathbf{F}=(\bar{Z}, D', \bar{\delta})$ of the automata $\mathbf{D}'_{y_1}, \mathbf{D}''_{y_1}, \dots, \mathbf{D}'_{y_k}, \mathbf{D}''_{y_k}$ with respect to $\bar{Z}$ and γ, where $\bar{Z}=Z_{y_1} \times Z_{y_1} \times \dots \times Z_{y_k} \times Z_{y_k}$ and γ is the identity mapping of $\bar{Z}$. Define a mapping $\varkappa: Y \to \bar{Z}$ with

$$\varkappa(y_i) = ((y_1)_e, (y_1)_e, \dots, (y_{i-1})_e, (y_{i-1})_e, \varkappa_{y_i}(y_i), (y_{i+1})_e, (y_{i+1})_e, \dots, (y_k)_e, (y_k)_e)$$

for all $y_i \in Y$. A short computation shows that the pair $\varkappa: Y \to \bar{Z}$ and $\psi\eta: B \to D'$ is an isomorphism of $\mathbf{B}$ into $\mathbf{F}$. Moreover, $\mathbf{F} \in \sum_3$, which ends the proof of Theorem 11.

Corollary. *A system $\sum$ of automata is homomorphically complete with respect to the (T, α_0)-product if and only if it is isomorphically complete with respect to the (T, α_0)-product.*

Now we are ready to present a stronger result. First we prove

Lemma 5. *Let $\mathbf{B}=(Y, B, \delta)$ be an automaton with $Y=\{y, y_e\}$ such that y_e induces the identity mapping of B. If for any $b \in B$, the cycle (b, y) in $\mathbf{B}$ is of type $(0, t)$, where $t=1$ or t is a power of r and r is a fixed prime number, then $\mathbf{B}$ can be represented isomorphically by an α_0-power of $\mathbf{C}_{(r)}$.*

Proof. Like in the proof of Lemma 4, B can be given as a union of pairwise disjoint subsets $B_1, \dots, B_k$ such that $\mathbf{B}_i=(Y, B_i, \delta_i)$ $(i=1, \dots, k)$ are connected subautomata of $\mathbf{B}$. By our assumption, $\mathbf{B}$ has no initial states. Therefore, every B_i

is a cycle of type $(0, t_i)$, where $t_i=1$ or r^l. For any $i(=1, \ldots, k)$ define the partitions $\pi_i(=\pi_{i0})$ as in Lemma 4.

Let us distinguish the following three cases:

(1) $t_i=k=1$. Then **B** is a one-state automaton. Obviously, it can be represented isomorphically by an α_0-power of $\mathbf{C}_{(r)}$ (having a single factor).

(2) $t_i=r^l$ and $k=1$. Then, by the proof of Lemma 4, **B** is an α_0-power of $\mathbf{C}_{(r)}$.

(3) $k>1$. If $t_i=1$ then $\mathbf{B}/\pi_i$ has two states and both input signals induce the identity mapping of its state set. Therefore, $\mathbf{B}/\pi_i$ is isomorphic to an α_0-power of $\mathbf{C}_{(r)}$ (having one factor only). Now if $t_i=r^l$ then $\mathbf{B}/\pi_i$ can be represented isomorphically by an α_0-power of $\mathbf{C}_{(r)}$ having $l+1$ factors. This can be proved in the same way as the corresponding statement in Lemma 4. The only difference is that here we need $\mathbf{C}_{(r)}$ instead of $\mathbf{E}_{(2)}$.

Since the intersection $\bigcap(\pi_i|i=1, \ldots, k)$ is the trivial partition of B having one-element blocks only, thus **B** can be represented isomorphically by an α_0-power of $\mathbf{C}_{(r)}$.

Theorem 12. *Let $\sum$ be a system of automata. An automaton* **B** *can be represented homomorphically by a (T, α_0)-product of automata from $\sum$ if and only if* **B** *can be represented isomorphically by a (T, α_0)-product of automata from $\sum$.*

Proof. Assume that $\mathbf{B}=(Y, B, \delta')$ can be represented homomorphically by a (T, α_0)-product of automata from $\sum$. If there are $b\in B$ and $y\in Y$ such that for the type (u, v) of the cycle (b, y) in **B** we have $u>0$ then, by the proof of the necessity of Theorem 10, there exist $\mathbf{A}=(X, A, \delta)\in\sum$, $a_1, a_2\in A$ and $p\in F(X)$ with $a_1\neq a_2$ and $a_1p=a_2p=a_2$. Therefore, by Theorem 11, $\sum$ is isomorphically complete with respect to the (T, α_0)-product.

Thus, we may assume that for all $b\in B$ and $y\in Y$ the cycles (b, y) in **B** are of type $(0, t)$. If $t=1$ for all cycles in **B** and $|B|>1$ then there exists an $\mathbf{A}\in\sum$ having at least two states. Obviously, **B** can be represented isomorphically by an α_0-power of **A**. Furthermore, it is also obvious that if $|B|=1$ then **B** can be represented isomorphically by an α_0-power of any automaton from $\sum$.

Now we can suppose that there exists at least one cycle (b, y) in **B** of type $(0, t)$ such that $t>1$. Moreover, it can also be assumed that $\sum$ is not homomorphically complete with respect to the (T, α_0)-product. Thus, there exist an $\mathbf{A}=(X', A, \delta)\in\sum$, $a\in A$ and $x'\in X'$ such that the cycle (a, x') is of type $(0, l)$ with $l>1$.

Let $Y=\{y_1, \ldots, y_s\}$, and denote by $\mathbf{B}_i=(Z_i, B, \delta_i)$ the automaton for which $Z_i=\{y_i, z_e\}$, $\delta_i(b, y_i)=\delta'(b, y_i)$ for all $b\in B$, and z_e induces the identity mapping of B. Every $\mathbf{B}_i$ can be given as a union of pairwise disjoint connected subautomata $\mathbf{B}_{ij}=(Z_i, B_{ij}, \delta_{ij})$ $(j=1, \ldots, m_i)$ such that each $\mathbf{B}_{ij}$ is y_i-cyclic of type $(0, t_{ij})$. Set $m=\max\{m_i|i=1, \ldots, s\}$ and $t=\max\{t_{ij}|i=1, \ldots, s;\ j=1, \ldots, m_i\}$. We show that there are automata $\mathbf{D}'_i=(Z_i, D_i, \delta'_i)$ and $\mathbf{D}''_i=(Z_i, D_i, \delta'')$ $(i=1, \ldots, s)$ in $\sum_2$ such that $\mathbf{B}_i$ is isomorphic to a subautomaton of a temporal product of $\mathbf{D}'_i$ by $\mathbf{D}''_i$.

For the sake of simplicity, assume that $m_i=u$ and $t_{ij}=v_j$. Moreover, let

$$B_{ij} = \{c_0^{(j)}, \ldots, c_{v_j-1}^{(j)}\} \quad \text{and} \quad \delta_{ij}(c_v^{(j)}, y_i) = c_{(v+1)\,(\mathrm{mod}\, v_j)}^{(j)}.$$

Take a prime r with $r|l$, and let w be a power of r such that $w \geqq 2t$. For every k ($k=1, \ldots, m$) define an automaton $\mathbf{C}_k=(Z_i, C_k, \bar{\delta}_k)$, where

$$C_k = \{d_0^{(k)}, \ldots, d_{w-1}^{(k)}\}, \quad \bar{\delta}_k(d_v^{(k)}, y_i) = d_{(v+1)\,(\mathrm{mod}\, w)}^{(k)}$$

and

$$\bar{\delta}_k(d_v^{(k)}, z_e') = d_v^{(k)} \quad \text{for all} \quad v = (0, \ldots, w-1).$$

Assume that these sets C_k are pairwise disjoint. Define D_i by $D_i = \bigcup (C_k | k=1, \ldots, m)$,

$$\delta_i'(d_v^{(k)}, z) = \bar{\delta}_k(d_v^{(k)}, z) \quad \text{for all} \quad z \in Z_i.$$

$\mathbf{D}_i''$ is defined similarly. It differs from $\mathbf{D}_i'$ only in that for all $j=1, \ldots, u$, if $w > 2v_j$ then

$$\delta_i''(d_{2v_j-1}^{(j)}, y_i) = d_0^{(j)}, \quad \delta_i''(d_0^{(j)}, y_i) = d_{2v_j}^{(j)}, \quad \delta_i''(d_v^{(j)}, y_i) = d_{v+1}^{(j)}$$

whenever $2v_j \leqq v < w-1$, and $\delta_i''(d_{w-1}^{(j)}) = d_1^{(j)}$. In all other cases the transitions are the same as in $\mathbf{D}_i'$. By Lemma 5, both $\mathbf{D}_i'$ and $\mathbf{D}_i''$ are in $\sum_2$, since $\mathbf{C}_{(r)}$ is isomorphic to a subautomaton of an α_0-power of $\mathbf{A}$.

Now take the temporal product $\mathbf{D}_i=(V_i, D_i, \delta_i^*)$ of $\mathbf{D}_i'$ by $\mathbf{D}_i''$ with respect to V_i and γ_i, where $V_i=Z_i \times Z_i$ and γ_i is the identity mapping of V_i. A routine computation shows that the pair of mappings $\varkappa_i$: $z \to (z, z)$ ($z \in Z_i$) and ψ_i: $c_v^{(j)} \to d_{2v}^{(j)}$ is an isomorphism of $\mathbf{B}_i$ into $\mathbf{D}_i$.

Observe that the cardinality of D_i is independent of i ($i=1, \ldots, s$). Therefore, by a suitable renaming of the elements of D_i we can achive that $D_1=\ldots=D_s$ $(=D)$ and $\psi_i(b)=\psi_j(b)$ for all $i, j=1, \ldots, s$. Using the same idea as in the proof of Theorem 11, one can show that $\mathbf{B}$ is isomorphic to a subautomaton of a temporal product of $\mathbf{D}_1', \mathbf{D}_1'', \ldots, \mathbf{D}_s', \mathbf{D}_s''$. This ends the proof of Theorem 12.

We say that an automaton $\mathbf{A}=(X, A, \delta)$ is *completely isolated* if $\delta(a, x)=a$ for any $a \in A$ and $x \in X$.

Theorem 13. *A set $\sum$ of automata is homomorphically complete with respect to the T-product or (T, α_i)-product ($i=1, 2, \ldots$) if and only if there is an automaton in $\sum$ which is not completely isolated.*

Proof. Since the products and temporal products of completely isolated automata are completely isolated thus the conditions of Theorem 13 are obviously necessary.

Conversely, assume that there exists an $\mathbf{A}=(X, A, \delta)$ in $\sum$ which is not completely isolated. Then the following two cases can occur:

(i) There are $a, b \in A$ and $p \in F(X)$ such that $a \neq b$ and $ap=bp=b$. Then, by Theorem 11, $\sum$ is isomorphically complete with respect to the (T, α_0)-product. Therefore, it is isomorphically complete with respect to the T-product or any (T, α_i)-product ($i=0, 1, \ldots$).

(ii) There are $p \in F(X)$, $x \in X$ and $a_0, \ldots, a_{t-1}$ $(t>1)$ such that $a_j \neq a_k$ if $j \neq k$ $(0 \leqq j, k < t)$, $a_j p = a_{(j+1)(\mathrm{mod}\, t)}$ and $\delta(a_j, x) = a_j$. Then the cyclic automaton $\mathbf{C}_{(t)}$ of type $(0, t)$ can be represented isomorphically by a temporal power of $\mathbf{A}$. Furthermore, it is obvious that the elevator $\mathbf{E}_{(2)}$ can be represented isomorphically by an α_1-power of $\mathbf{C}_{(t)}$. Therefore, since the α_0-product of α_1-products is an α_1-product, thus, by Theorem 11, we get that $\sum$ is isomorphically complete with respect to the (T, α_1)-product. This completes the proof of Theorem 13.

From the proof of Theorem 13 we get the following

Corollary. *A set $\sum$ of automata is homomorphically complete with respect to the T-product or (T, α_i)-products $(i>0)$ if and only if it is isomorphically complete with respect to the T-product or (T, α_i)-products with $i>0$.*

References

[1] F. Gécseg and I. Peák, *Algebraic theory of automata,* Akadémiai Kiadó (Budapest, 1972).

[2] F. Gécseg and S. Székely, On equational classes of unoids, *Acta Sci. Math.*, **34** (1973), 99—101.

[3] F. Gécseg, Composition of automata, *Proceedings of the 2nd Colloquium on Automata, Languages and Programming,* Saarbrücken, 1974, Springer Lecture Notes in Computer Science, Vol. 14, 351—363.

[4] F. Gécseg, On subdirect representations of finite commutative unoids, *Acta Sci. Math.*, **36** (1974), 33—38.

[5] A. Gill, Single-channel and multichannel finite-state machines, *IEEE Trans. Comput.*, **19** (1970) N°-11, 1073—1078.

[6] A. Ginzburg, *Algebraic theory of automata*, Academic Press (New York—London, 1968).

[7] Глушков, В. М. О полноте систем операции в электронных вычислительных машин, *Кибернетика,* **2** (1968), 1—5.

[8] Иванов, Г. И., Временные преобразования цифровых автоматов, *Изв. Акад. Наук. СССР, Техническая кибернетика,* **6** (1973), 106—113.

[9] Летичевский, А. А., Условия полноты для конечных автоматов, *Журн. вычисл. матем. и матем. физ.*, **1** (1961), 702—710.

BOLYAI INSTITUTE
ARADI VÉRTANUK TERE 1
6720 SZEGED, HUNGARY