Acta Sci. Math., 38 (1976), 69—72

On degree of approximation of a class of functions by means of Fourier series

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1. Let f be periodic with period 2π, and integrable in the Lebesgue sense. The Fourier series associated with f at the point x, is given by

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \tag{1.1}$$

If $\{p_n\}$ is a sequence of positive constants, such that

$$P_n = p_0 + p_1 + \ldots + p_n \to \infty \quad \text{as} \quad n \to \infty$$

then a given series $\sum_{n=0}^{\infty} c_n$ with the sequence of partial sums $\{s_n\}$ is said to be Nörlund summable (N, p_n) to s, provided that

$$T_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} s_k \to s \quad \text{as} \quad n \to \infty.$$

We call T_n the (N, p_n)-mean or Nörlund mean of $\sum c_n$. In the following we assume that the Nörlund means are regular, more precisely, we assume that

$$0 < np_n \leqq cP_n \quad \text{for} \quad n = 1, 2, \ldots, \text{ and } p_0 > 0. \tag{1.2}$$

2. The following theorem on the degree of approximation of a function $f \in \text{Lip}\, \alpha$, by the (C, δ)-means of its Fourier series, is due to G. ALEXITS [1].

Theorem A. *If a periodic function $f \in Lip\, \alpha$ for $0 < \alpha \leqq 1$, then the degree of approximation of the (C, δ)-means of its Fourier series for $0 < \alpha < \delta \leqq 1$ is given by*

$$\max_{0 \leqq x \leqq 2\pi} |f(x) - \sigma_n^{(\delta)}(x)| = O\left(\frac{1}{n^{\alpha}}\right)$$

Received July 20, 1975.

and for $0<\alpha\leqq\delta\leqq 1$, *is given by*

$$\max_{0\leqq x\leqq 2\pi}|f(x)-\sigma_n^{(\delta)}(x)| = O\left(\frac{\log n}{n^\alpha}\right)$$

where $\sigma_n^{(\delta)}$ *are the* (C, δ)*-means of the partial sums of* (1.1).

Let $C^*[0, 2\pi]$ denote the class of all continuous functions on $[0, 2\pi]$, periodic and of period 2π. The object of this paper is to prove the following theorem.

Theorem. *If* $\omega(t)$ *is the modulus of continuity of* $f\in C^*[0, 2\pi]$, *then the degree of approximation of* f *by the Nörlund means of the Fourier series for* f *is given by*

$$E_n \equiv \max_{0\leqq t\leqq 2\pi}|f(t)-T_n(t)| = O\left\{\frac{1}{P_n}\sum_{k=1}^{n}\frac{P_k\omega(1/k)}{k}\right\},$$

where T_n *are the* (N, p_n)*-means of the Fourier series for* f.

If we deal with Cesàro means of order δ and consider a function $f\in\operatorname{Lip}\alpha$, $0<\alpha\leqq 1$, then our Theorem reduces to Theorem A.

Proof.

$$T_n(x)-f(x) = \frac{1}{2\pi P_n}\int_0^\pi \{f(x+t)+f(x-t)-2f(x)\}\sum_{k=0}^{n} p_{n-k}\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\,dt.$$

If we write $\varphi(t)=|\frac{1}{2}f(x+t)+\frac{1}{2}f(x-t)-f(x)|$ then it is clear that

$$\varphi(t)\leqq\omega(t),$$

and therefore,

$$|f(x)-T_n(x)| \leqq \frac{1}{\pi P_n}\int_0^{\pi/n}\frac{\omega(t)}{t/2}\left|\sum_{k=0}^{n} p_{n-k}\sin(k+1/2)t\right|dt+$$

$$+\frac{1}{\pi P_n}\int_{\pi/n}^{\pi}\frac{\omega(t)}{t/2}\left|\sum_{k=0}^{n}p_{n-k}\sin kt\right|dt+\frac{1}{\pi P_n}\int_{\pi/n}^{\pi}\frac{\omega(t)}{t/2}\left|\sum_{k=0}^{n}p_{n-k}\cos kt\right|dt =$$

$$= I_1+I_2+I_3,\ \text{say}.$$

Now

$$I_1 = \frac{1}{\pi P_n}\int_0^{\pi/n}\frac{\omega(t)}{t/2}\left|\sum_{k=0}^{n}p_{n-k}\sin(k+1/2)t\right|dt =$$

$$= O\left(\frac{1}{P_n}\right)\int_0^{\pi/n}\frac{\omega(t)}{t}\sum_{k=0}^{n}p_{n-k}(k+1/2)t\,dt =$$

$$= O\left(\frac{1}{P_n}\right)\int_0^{\pi/n}\omega(t)\,dt\sum_{k=0}^{n}p_{n-k}(k+1/2) = O\left(\frac{1}{nP_n}\right)\omega\left(\frac{1}{n}\right)\sum_{k=0}^{n}kp_{n-k} = O(\omega(1/n)).$$

By (1.2),

$$\frac{1}{P_n}\sum_{k=0}^{n}\frac{P_k\omega(1/k)}{k} \geqq \frac{\omega(1/n)}{cP_n}\sum_{k=0}^{n}p_k = \frac{1}{c}\omega(1/n),$$

consequently,

$$I_1 = O\left\{\frac{1}{P_n}\sum_{k=0}^{n}\frac{P_k\omega(1/k)}{k}\right\}.$$

Now

$$I_2 \leqq \frac{2}{\pi P_n}\int_{\pi/n}^{\pi}\frac{\omega(t)}{t}\left|\sum_{k=0}^{n}p_{n-k}\sin kt\right|dt = O\left\{\frac{1}{P_n}\int_{\pi/n}^{\pi}\frac{\omega(t)}{t}P\left(\frac{1}{t}\right)dt\right\} =$$

$$= O\left\{\frac{1}{P_n}\int_{n/\pi}^{1/\pi}\frac{\omega(1/t)}{1/t}P(t)\left(-\frac{dt}{t^2}\right)\right\} = O\left\{\frac{1}{P_n}\int_{n/\pi}^{1/\pi}\frac{\omega(1/t)}{t}P(t)\,dt\right\} =$$

$$= O\left\{\frac{1}{P_n}\sum_{k=1}^{n}\frac{P(k)\omega(1/k)}{k}\right\}, \quad \text{where} \quad P(k) = P_{[k]}.$$

Similarly,

$$I_3 \leqq \frac{2}{\pi P_n}\int_{\pi/n}^{\pi}\omega(t)\,P\left(\frac{1}{t}\right)dt = O\left\{\frac{1}{P_n}\sum_{k=1}^{n}\frac{P(k)\omega(1/k)}{k^2}\right\}$$

which is dominated by the bound for I_2.

Adding the bounds for I_1, I_2, I_3 we have the desired result.

3. Remarks. It may be interesting to know the answers to the following questions:

(i) Can our Theorem be extended to matrix summability?

(ii) Can the result be extended to differentiated Fourier series?

(iii) Can the result be extended to some other series, viz. Legendre series, ultraspherical, Bessel series, etc?

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