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Unitary dilations and C*-algebras

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The purpose of this Note is to present a "global", i.e., point-free characterization of the $\mathscr{C}_a$-classes of operators introduced as generalizations of the $\mathscr{C}_\varrho$-classes of SZ.-NAGY and FOIAŞ [2] (for the details see **1**). This permits us to define analogous classes in an arbitrary C^*-algebra. Certain properties of the $\mathscr{C}_a$-classes derive in a simpler way in this more general setting.

1.

Let H be a complex Hilbert space, $B(H)$ the C^*-algebra of all bounded linear operators of H. Denote by $B^+(H)$ the convex cone of positive elements of $B(H)$. Consider a boundedly invertible element a of $B^+(H)$. An element $\mathbf{x}$ of $B(H)$ is said to admit a *unitary a-dilation* if there exists a Hilbert space K containing H as its subspace and a unitary element $\mathbf{u}$ of $B(K)$ such that

$$a^{-1/2}x^n a^{-1/2} = \mathrm{pr}_H u^n \quad (n = 1, 2, \ldots).$$

The set of elements of $B(H)$ which admit unitary a-dilations is denoted by $\mathscr{C}_a$. LANGER characterized the $\mathscr{C}_a$-classes in the following manner (cf. [3], pp. 53—54):

An element x of $B(H)$ belongs to $\mathscr{C}_a$ if and only if:

(i) *the spectrum $\sigma(x)$ of x is contained in the closed disc $\mathbf{C}_1$ of the complex number field $\mathbf{C}$, and*

(ii) *for every $\xi \in H$ and $\mu \in \mathbf{C}_1$,*

$$\langle a\xi, \xi\rangle - 2\,\mathrm{Re}\,\langle \mu(a-e)x\xi, \xi\rangle + |\mu|^2\langle (a-2e)x\xi, x\xi\rangle \geqq 0$$

(e denotes the identity operator of H).

Since

$$\mathrm{Re}\,\langle y\xi, \xi\rangle = \langle (\mathrm{Re}\, y)\xi, \xi\rangle, \qquad (y \in B(H),\ \xi \in H)$$

the preceding inequality can be written as

$$\langle (a - 2\,\mathrm{Re}\,\mu(a-e)x + |\mu|^2 x^*(a-2e)x)\xi, \xi\rangle \geqq 0,$$

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i.e.,

$$a-2\operatorname{Re}\mu(a-e)x+|\mu|^2x^*(a-2e)x \geqq 0. \tag{1}$$

But

$$\begin{aligned} a-2\operatorname{Re}\mu(a-e)x+|\mu|^2x^*(a-2e)x &= \\ = a-\mu ax-\bar{\mu}x^*a+\mu x+\bar{\mu}x^*+(\mu x)^*a(\mu x)-(\mu x)^*(\mu x)-|\mu|^2x^*x+e-e &= \\ = [(\mu x)^*a-a-(\mu x)^*+e](\mu x)-[(\mu x)^*a-a-(\mu x)^*+e]+e-|\mu|^2x^*x &= \\ = (\mu x-e)^*(a-e)(\mu x-e)+e-|\mu|^2x^*x. \end{aligned}$$

Thus (1) is equivalent to

$$|\mu|^2x^*x \leqq e+(\mu x-e)^*(a-e)(\mu x-e)$$

or, putting $\mu=1/\lambda$, to

$$x^*x \leqq |\lambda|^2e+(x-\lambda e)^*(a-e)(x-\lambda e).$$

Thus condition (ii) is equivalent to condition

$$x^*x \leqq |\lambda|^2e+(x-\lambda e)^*(a-e)(x-\lambda e) \quad \textit{for all} \quad \lambda\in\mathbf{C},\ |\lambda|\geqq 1. \tag{iii}$$

Summing up the results, we obtain

Proposition 1. *Let $a\in B^+(H)$ be arbitrary. For an element x of $B(H)$ conditions* (ii) *and* (iii) *are equivalent.*

2.

Let A be an arbitrary complex C^*-algebra with unity e. Denote by A^+ the convex cone of positive elements of A. Let $a\in A^+$ be arbitrary. Denote by C_a the set of the elements x of A which satisfy condition (iii).

Proposition 2. *C_a is an increasing function of a in the sense that $a_1, a_2\in A^+$, $a_1\leqq a_2$ imply $C_{a_2}\subseteqq C_{a_1}$.*

Proof. This is a consequence of the fact that for every $y\in A$ we have $y^*a_1y\leqq \leqq y^*a_2y$.

Proposition 3. *If $\|a\|<2$, then for $x\in C_a$ we have*

$$\|x\| \leqq (\|a\|/(2-\|a\|))^{1/2}. \tag{2}$$

In particular, $\|a\|<1$ implies $\|x\|<1$ for every $x\in C_a$.

Proof. For $\lambda=\pm 1$, condition (iii) takes the forms

$$x^*x \leqq e+(x-e)^*(a-e)(x-e), \quad x^*x \leqq e+(x+e)^*(a-e)(x+e).$$

By adding up these two inequalities, we obtain

$$2x^*x \leqq x^*ax+a.$$

Now, it is known that in a C^*-algebra $u\geqq 0$, $v\geqq 0$, $u\leqq v$ imply $\|u\|\leqq\|v\|$. Hence,

$$2\|x^*x\| = 2\|x\|^2 \leqq \|x\|^2\|a\|+\|a\|,$$

i.e.

$$(2-\|a\|)\|x\|^2 \leqq \|a\|,$$

which is equivalent to (2). The rest of the proof is obvious.

Theorem 1. *If $x\in C_a$, then the spectrum $\sigma(x)$ of x is contained in* $\mathbf{C}_1$.

Proof. We know that $\sigma(x)$ is the union of the left-spectrum $\sigma_l(x)$ and the right-spectrum $\sigma_r(x)$ of x. Furthermore,

$$\sup_{\lambda\in\sigma_l(x)}|\lambda| = \sup_{\lambda\in\sigma_r(x)}|\lambda|.$$

Thus, it suffices to show that, for instance, we have $\sigma_l(x)\subset\mathbf{C}_1$. Let S_x^l be the set of all left x-multiplicative states of A: $s\in S_x^l$ if s is a state and if $s(yx)=s(y)s(x)$ for all $y\in A$. It is known ([1]) that

$$\sigma_l(x) = \{s(x): s\in S_x^l\}.$$

Let μ now be an arbitrary element of $\sigma_l(x)$ and s an element of S_x^l for which $\mu=s(x)$. Apply s to the inequality in (iii). We get

$$s(x^*x) = s(x^*)s(x) = |\mu|^2 \leqq |\lambda|^2+|\mu-\lambda|^2(s(a)-1) \tag{3}$$

for each $\lambda\in\mathbf{C}$, $|\lambda|\geqq 1$. Assume that $|\mu|>1$ and consider a real number ϱ such that $0<\varrho<1-1/|\mu|$. Put $\lambda=\mu-\varrho\mu$. Then $|\lambda|>1$. For this particular λ, relation (3) gives

$$|\mu|^2 \leqq |\mu|^2(s-\varrho)^2+\varrho^2|\mu|^2(s(a)-1) = |\mu|^2-2\varrho|\mu|^2+\varrho^2|\mu|^2 s(a).$$

This leads us to the inequality $0\leqq -2+\varrho s(a)$. If we let ϱ tend to zero, we obtain $0\leqq -2$: a contradiction. Thus Theorem 1 is proved.

Consider now the case $A=B(H)$. Theorem 1 allows us to formulate the following "global" characterization of the $\mathscr{C}_a$ classes.

Theorem 2. *An element x of $B(H)$ belongs to $\mathscr{C}_a$ if and only if it satisfies condition* (iii).

Proof. On account of Langer's result mentioned in **1**, the necessity part of the theorem follows from Proposition 1. The sufficiency part is a consequence of Theorem 1 and Proposition 1 (using Langer's result).

Corollary 1. *$\mathscr{C}_a$ is an increasing function of a.*

Corollary 2. *If* $\|a\|<1$, *then the minimal unitary dilation of every element* x *of* $\mathscr{C}_a$ *is a bilateral shift with multiplicity equal to* dim H.

Proof. See proposition 3 of the present paper and [3], Cor. II. 7. 5.

Problem. We could not decide yet whether C_a is a strictly increasing function of a or not.

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