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An integrability theorem for power series

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In [6] we proved the following

Theorem A. *Let $\lambda(t)>0$ be a nonincreassing, integrable function on the interval $0<t\leqq 1$ such that $\lambda(1/n+1)=O(\lambda(1/n))$, and let $A(x)$ be defined on the interval $0\leqq x<1$ by the series $\sum_{k=0}^{\infty} a_k x^k$ with $a_k\geqq 0$. Furthermore let $0<p\leqq\infty$. Then $\lambda(1-x)(A(x))^p\in L(0,1)$ if and only if*

$$\sum_{n=1}^{\infty}\lambda(1/n)n^{-2}\left(\sum_{k=0}^{n}a_k\right)^p<\infty. \tag{1}$$

If $\lambda(t)=t^{-\gamma}(\gamma<1)$, Theorem A reduces to a theorem of KHAN [5], which in its turn includes a theorem of ASKEY ([1], $\gamma=0$) and a theorem of HEYWOOD ([2], $p=1$).

In [6], Theorem A was stated for $p\geqq 1$ only, but it is easy to see that the proof actually holds for $0<p<1$, too.

Recently JAIN [4] obtained

Theorem B. *Let*

$$B(x)=\sum_{n=0}^{\infty}b_n x^n,\quad 0\leqq x\leqq 1\quad \textit{and}\quad \gamma<1.$$

Suppose that there is a positive number ε such that

$$b_n>\frac{-K}{n^{(\gamma/p)+1+\varepsilon-1/p}}\quad (0<p<\infty,\ K\ \textit{constant})$$

for all sufficiently large values of n. Then

$$(1-x)^{-\gamma}|B(x)|^p\in L(0,1)$$

if and only if

$$\sum_{n=1}^{\infty}n^{\gamma-2}\left(\sum_{k=1}^{n}|c_k|\right)^p<\infty.$$

In the particular case $p=1$ Theorem B was proved by HEYWOOD [3].

In the present paper Theorem B will be generalized as follows:

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Theorem. *Let $\lambda(t)>0$ be a nonincreasing function on the interval $0<t\leqq 1$ such that*

$$\sum_{n=k}^{\infty}\lambda(1/n)n^{-2}\leqq M\lambda(1/k)/k \tag{2}$$

and let

$$F(x)=\sum_{n=0}^{\infty}c_n x^n,\quad 0\leqq x<1.$$

Suppose there is a positive monotonic sequence $\{\varrho_n\}$ with $\sum_{n=1}^{\infty}1/n\varrho_n<\infty$ such that

$$c_n>\frac{-K}{(\varrho_n\lambda(1/n))^{1/p}\cdot n^{1-1/p}}\quad (0<p<\infty,\ K>0) \tag{3}$$

for all sufficiently large values of n. Then $\lambda(1-x)|F(x)|^p\in L(0,1)$ if and only if

$$\sum_{n=1}^{\infty}\lambda(1/n)n^{-2}\left(\sum_{k=0}^{n}|c_k|\right)^p<\infty. \tag{4}$$

It is clear that if $\lambda(t)=t^{-\gamma}$ $(\gamma<1)$ then our Theorem reduces to Theorem B.

Proof. Let

$$A(x)=\sum_{n=0}^{\infty}a_n x^n\quad\text{for}\quad 0\leqq x<1$$

with $a_0=0$ and

$$a_n=\frac{K}{(\varrho_n\lambda(1/n))^{1/p}n^{1-1/p}}\quad\text{for}\quad n\geqq 1.$$

First we show that these coefficients a_n satisfy condition (1). If $p\geqq 1$ then we use the inequality

$$\sum_{n=1}^{\infty}\lambda_n\left(\sum_{k=1}^{n}a_k\right)^p\leqq p^p\sum_{n=1}^{\infty}\lambda_n^{1-p}\left(\sum_{k=n}^{\infty}\lambda_k\right)^p a_n^p, \tag{5}$$

which holds for any $\lambda_n>0$ and $a_n\geqq 0$ (see [7], inequality (1′)), with $\lambda_n=\lambda(1/n)n^{-2}$. Using (5), by (2), we have

$$\sum_{n=1}^{\infty}\lambda(1/n)n^{-2}\left(\sum_{k=1}^{n}a_k\right)^p\leqq O(1)\sum_{n=1}^{\infty}\lambda(1/n)n^{-2+p}a_n^p\leqq$$

$$\leqq O(1)\sum_{n=1}^{\infty}\lambda(1/n)n^{-2+p}(\varrho_n\lambda(1/n)n^{p-1})^{-1}\leqq O(1)\sum_{n=1}^{\infty}1/n\varrho_n<\infty.$$

If $0<p<1$, using some elementary estimates and (2), we obtain

$$\sum_{n=2}^{\infty}\lambda(1/n)n^{-2}\left(\sum_{k=1}^{n}a_k\right)^p\leqq\sum_{m=0}^{\infty}\sum_{n=2^m+1}^{2^{m+1}}\lambda(1/n)n^{-2}\left(\sum_{a=1}^{2^{m+1}}a_k\right)^p\leqq$$

$$\leqq O(1)\sum_{m=0}^{\infty}\lambda(1/2^{m+1})2^{-m}\left(\sum_{k=1}^{m+1}(2^k)^{1/p}(\lambda(1/2^k)\varrho_{2^k})^{-1/p}\right)^p \leqq$$

$$\leqq O(1)\sum_{k=1}^{\infty}(2^k/\varrho_{2^k}\lambda(1/2^k))\sum_{m=k}^{\infty}\lambda(1/2^m)2^{-m} \leqq O(1)\sum_{k=1}^{\infty}1/\varrho_{2^k} < \infty.$$

Hereby we proved that the coefficients of the function $A(x)$ satisfy condition (1), so by Theorem A

$$\lambda(1-x)(A(x))^p \in L(0,1). \tag{6}$$

By (3) the coefficients a_n+c_n are positive for all sufficiently large values of n, thus the function

$$A(x)+F(x) = \sum_{n=0}^{\infty}(a_n+c_n)x^n$$

has the property

$$\lambda(1-x)(A(x)+F(x))^p \in L(0,1) \tag{7}$$

if and only if

$$\sum_{n=1}^{\infty}\lambda(1/n)n^{-2}\left(\sum_{k=0}^{n}(a_k+c_k)\right)^p < \infty. \tag{8}$$

Hence we obtain the statement of Theorem easily. Indeed, if $\lambda(1-x)|F(x)|^p \in \in L(0,1)$ then (6) implies (7), which implies (8). But by (3) we have

$$|c_n| \leqq 2a_n + c_n$$

whence, by (8), (4) follows. If (4) holds, then this implies (8) and equivalently (7). From (6) and (7), $\lambda(1-x)|F(x)|^p \in L(0,1)$ follows obviously.

Thus Theorem is proved.

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