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Generalization of a converse of Hölder's inequality

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In [1] we proved the integral inequality

$$\int_{-\infty}^{\infty} \sup_{\sum_{i=1}^{n} x_i = t} \prod_{i=1}^{n} f_i(x_i)\, dt \geqq \prod_{i=1}^{n} (p_i)^{1/p_i} \Big(\int_{-\infty}^{\infty} f_i^{p_i}(x_i)\, dx_i \Big)^{1/p_i} \tag{1}$$

for nonnegative step functions $f_i(x_i)$ $(i=1, 2, \ldots, n)$ and exponents p_i satisfying the conditions $1 \leqq p_i \leqq \infty$ and $\sum_{i=1}^{n} 1/p_i = 1$.

In the course of the proof of (1) we implicitly also proved the inequality

$$\int_{-\infty}^{\infty} \sup_{\sum_{i=1}^{n} x_i = t} \prod_{i=1}^{n} F_i(x_i)\, dt \geqq \sum_{i=1}^{n} \int_{-\infty}^{\infty} F_i^{p_i}(x_i)\, dx_i, \tag{2}$$

where

$$F_i(x_i) = (\max f_i)^{-1} f_i(x).$$

In the present paper inequality (2) will be generalized in two directions.

Let H_n denote the set of nonnegative and continuous functions $H(x_1, x_2, \ldots, x_n)$ of n variables such that $H(0, 0, \ldots, 0) = 0$ and

$$H(x_1, \ldots, x_n) \geqq \min(|x_1|, |x_2|, \ldots, |x|_n \quad \text{at any point} \quad (x_1, x_2, \ldots, x_n). \tag{3}$$

Furthermore, let $S(M)$ denote the set of nonnegative step functions $f(x)$ with $\max_x f(x) = M$.

We prove the following

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Theorem. *Suppose $H(x_1, \ldots, x_n) \in H_n$ and $f_i(x) \in S(M)$ $(i=1, 2, \ldots, n)$. Then we have for any $\Delta \geqq 0$*

$$\int_{-\infty}^{\infty} \sup_{t \leqq \sum_{i=1}^{n} x_i \leqq t+\Delta} H(f_1(x_1), f_2(x_2), \ldots, f_n(x_n))\, dt \geqq \tag{4}$$

$$\geqq \sum_{i=1}^{n} \int_{-\infty}^{\infty} f_i(x)\, dx + \Delta \cdot \max_{x_1, x_2, \ldots, x_n} H(f_1(x_1), \ldots, f_n(x_n)).$$

In the particular case $\Delta = 0$ and $H(x_1, \ldots, x_n) = \prod_{i=1}^{n} |x_i|^{1/p_i}$ we obtain inequality (1) by replacing $f_i(x)$ by $f_i^{p_i}(x)$ and using the well-known inequality

$$\prod_{i=1}^{n} \varrho_i \leqq \sum_{i=1}^{n} \frac{1}{p_i} (\varrho_i)^{p_i} \quad \text{for} \quad \varrho_i \geqq 0, \quad \sum_{i=1}^{n} \frac{1}{p_i} = 1.$$

Next we remark that if one of the functions $f_i(x)$ belongs to $S(M')$, where $M' \neq M$, then inequality (4) does not necessarily hold.

Finally we mention that from our theorem we can deduce an inequality concerning series of positive terms.

Let $s^+(M)$ denote the set of sequences $a = \{a_n\}$ with $a_n \geqq 0$ and $\max_n a_n = M$. Furthermore let

$$\|a\|_\infty = \sup_n a_n \quad \text{and} \quad \|a\|_p = \left\{ \sum_{n=-\infty}^{\infty} a_n^p \right\}^{1/p}.$$

Corollary. *Suppose $H(x_1, \ldots, x_n) \in H_n$ and $a^{(i)} \in s^+(M)$ $(i=1, 2, \ldots, n)$. Then*

$$(n-1) \sup_{k_1, \ldots, k_n} H(a_{k_1}^{(1)}, a_{k_2}^{(2)}, \ldots, a_{k_n}^{(n)}) + \sum_{k=-\infty}^{\infty} \sup_{k \leqq k_1+k_2+\ldots+k_n \leqq k+l} H(a_{k_1}^{(1)}, \ldots, a_{k_n}^{(n)}) \geqq \tag{5}$$

$$\geqq \sum_{i=1}^{n} \sum_{k=-\infty}^{n} a_k^{(i)} + l \sup_{k_1, \ldots, k_n} H(a_{k_1}^{(1)}, \ldots, a_{k_n}^{(n)})$$

holds for any nonnegative integer l.

Hence, taking $H(x_1, \ldots, x_n) = \prod_{i=1}^{n} |x_i|^{1/p_i}$ $\left(\sum_{i=1}^{n} 1/p_i = 1 \right)$ and replacing $a_k^{(i)}$ by $(a_k^{(i)})^{p_i}$ we obtain the inequality

$$\sum_{k=-\infty}^{\infty} \sup_{k \leqq k_1+\ldots+k_n \leqq k+l} a_{k_1}^{(1)} a_{k_2}^{(2)} \ldots a_{k_n}^{(n)} \geqq \prod_{i=1}^{n} \|a^{(i)}\|_\infty \left\{ \sum_{i=1}^{n} \|a^{(i)}\|_\infty^{-p_i} \|a^{(i)}\|_{p_i}^{p_i} + l - n + 1 \right\}, \tag{6}$$

where $a^{(i)}$ denotes an arbitrary nonnegative sequence. Inequality (6) was proved by B. Uhrin [3], the special case $l=0$ of (6) can be found in [2], too.

Proof of the theorem. The way of our proof is similar to the proof given by us in [1]. We may assume that the step functions $f_i(x)$ have integer points of discontinuity and have at their points of discontinuity the larger one of the values taken on the adjoining intervals (this convention will be of technical importance).

Let N be an integer such that if $|x|>N$ then $f_i(x)=0$ for all i; furthermore let

$$f_i(x) = a_k^i \quad \text{if} \quad x \in (k-1, k), \quad k = -N+1, -N+2, \ldots, N-1, N.$$

Let v_i denote a fixed index for which $a_{v_i}^i = M$. Furthermore we define the following auxiliary function:

$$F_i(x) = \begin{cases} f_i(x) & \text{if} \quad x \notin (v_i-1, v_i), \\ M+1 & \text{if} \quad x \in [v_i-1, v_i]. \end{cases}$$

It is clear that if b_k^i denote the values of $F_i(x)$ then $b_k^i = a_k^i$ if $k \neq v_i$ and $b_{v_i}^i = M+1$.

By means of these functions $F_i(x)$ we shall give a decomposition of the interval $(-\infty < t < \infty)$ such that the sum of the lower estimations to be given on the subintervals for the left-hand side of (4) be already greater than the right-hand side of (4).

First we consider the special case $\Delta = 0$.

By the definition of N we have

$$S \equiv \int_{-\infty}^{\infty} \sup_{\sum_{i=1}^{n} x_i = t} H(f_1(x_1), f_2(x_2), \ldots, f_n(x_n))\, dt =$$

$$= \int_{-nN}^{nN} \sup_{\sum_{i=1}^{n} x_i = t} H(f_1(x_1), f_2(x_2), \ldots, f_n(x_n))\, dt \equiv S_N,$$

thus it is enough to decompose the interval $[-nN, nN]$.

Let

$$s(t) = \begin{cases} 1 & \text{if} \quad t \geqq 0, \\ 0 & \text{if} \quad t < 0, \end{cases}$$

and we denote, as usual, by $h_+(u_0)$ the limit from the right of the function $h(u)$ at u_0, and by $h_-(u_0)$ its limit from the left. We put

$$P_0(y_1^0, y_2^0, \ldots, y_n^0) \equiv (-N, -N, \ldots, -N)$$

and define, for $m \geqq 1$, the following numbers and points successively:

$$u_i^m = s\big(\min_{j \neq i} F_{j+}(y_j^{m-1}) - F_{i+}(y_i^{m-1})\big)$$

and

$$P_m(y_1^m, y_2^m, \ldots, y_n^m) = (y_1^{m-1} + u_1^m, y_2^{m-1} + u_2^m, \ldots, y_n^{m-1} + u_n^m).$$

By the definition of the points P_m it is clear that starting from the point P_0 we go from a point P_m one step on the axis x_i where the minimum of the values $F_{i+}(y_i^m)$ ($i=$

$=1, 2, \ldots, n)$ is taken if it is reached only at one j; otherwise we go simultaneously one-one step on all of the axes where the value of $F_{i+}(y_i^m)$ equals the minimum value. We continue this procedure till $y_i^{m_0}=v_i$ will hold for some $m=m_0$ and for all i, i.e.

$$P_{m_0}(y_1^{m_0}, y_2^{m_0}, \ldots, y_n^{m0}) \equiv (v_1, v_2, \ldots, v_n).$$

This necessarily follows because of the definition of the functions $F_i(x)$ on the stripes $[v_i+1, v_i]$.

Then we define a sequence of points $Q_m(z_1^m, z_2^m, \ldots, z_n^m)$ in an analogous way comming back from the point $Q_0(z_1^0, z_2^0, \ldots, z_n^0)\equiv(N, N, \ldots, N)$. Similarly as before, we define, for $m\geqq 1$, the following numbers and points successively:

$$v_i^{(m)} = s\big(\min_{j\neq i} F_{j-}(z_j^{m-1})-F_{i-}(z_i^{m-1})\big)$$

and

$$Q_m(z_1^m, z_2^m, \ldots, z_n^m) = (z_1^{m-1}-v_1^m, z_2^{m-1}-v_2^m, \ldots, z_n^{m-1}-v_n^m).$$

For similar reasons as in the case of the points P_m, we come in a finite number, say m_1, steps to the point $P_{m_0}=(v_1, v_2, \ldots, v_n)$, i.e. $P_{m_0}=Q_{m_1}$. Now we can give a path going from the point P_0 to the point Q_0 such that by means of the "break points" of this path the required decomposition of the interval $[-nN<t<nN]$ can be given.

For each i $(i=1, 2, \ldots, n)$ we put

$$y_i^{m_0+l} = z_i^{m_1-l} \quad (l = 0, 1, \ldots, m_1);$$

hereby we arranged the points in a sequence $P_m(y_1^m, y_2^m, \ldots, y_n^m)$ $(m=0, 1, \ldots, m_0+m_1)$, which gives the required path from P_0 to Q_0.

Next we give the required decomposition of the interval $[-nN, nN]$. First we set for each i $(i=1, 2, \ldots, n)$

(7) $$I_i^m = y_i^m - y_i^{m-1} \quad (m = 1, 2, \ldots, m_0+m_1),$$

furthermore denote by c_i^m the value of $f_i(x_i)$ on the interval (y_i^{m-1}, y_i^m) if $I_i^m=1$, and at the point $x_i=y_i^m$ if $I_i^m=0$.

Let

(8) $$t_k = \sum_{i=1}^{n} y_i^k \quad (k = 0, 1, \ldots, m_0+m_1).$$

It is easy to see that $t_0= -nN$ and $t_{m_0+m_1}=nN$, furthermore for any $k\geqq 1$

$$t_k = t_{k-1}+t_k-t_{k-1} = t_{k-1}+\sum_{i=1}^{n} I_i^k.$$

Thus we can decompose each interval $[t_{k-1}, {}_k]$ by the points

(9) $$\tau_{k,0} = t_{k-1} \quad \text{and} \quad \tau_{k,j} = t_{k-1}+\sum_{i=1}^{j} I_i^k \quad (j = 1, 2, \ldots, n)$$

into subintervals. On such a subinterval $[\tau_{k,j-1}, \tau_{k,j}]$ we have for any k and j ($k=1, 2, \ldots, m_0+m_1$; $j=1, 2, \ldots, n$) the following lower estimate:

$$S_{k,j} \equiv \int_{\tau_{k,j-1}}^{\tau_{k,j}} \sup_{\sum_{i=1}^{n} x_i = t} H(f_1(x_1), \ldots, f_n(x_n))\, dt \geqq I_j^k c_j^k. \tag{10}$$

To verify this inequality we put $x_i = y_i^k$ for $i<j$ and $x_i = y_i^{k-1}$ for $i>j$, and let x_j run from y_j^{k-1} to y_j^k, then t goes from $\tau_{k,j-1}$ to $\tau_{k,j}$; in fact we have then, by (7), (8) and (9)

$$t = \sum_{i=1}^{n} x_i \geqq \sum_{i=1}^{j-1} y_i^k + \sum_{i=j}^{n} y_i^{k-1} = t_{k-1} + \sum_{i=1}^{j-1} I_i^k = \tau_{k,j-1}$$

and

$$t = \sum_{i=1}^{n} x_i \leqq \sum_{i=1}^{j} y_i^k + \sum_{i=j+1}^{n} y_i^{k-1} = t_{k-1} + \sum_{i=1}^{j} I_i^k = \tau_{k,j}.$$

Choosing the values of x_i as above and taking into account that I_j^k differs from zero only for such subscripts j for which $c_j^k \leqq c_i^k$ holds for all i ($i=1, 2, \ldots, n$), we obtain by (3) inequality (10) immediately.

By (9) and (10),

$$\sigma_k = \sum_{j=1}^{n} S_{k,j} = \int_{t_{k-1}}^{t_k} \sup_{\sum_{i=1}^{n} x_i = t} H(f_1(x_1), \ldots, f_n(x_n))\, dt \geqq \sum_{j=1}^{n} I_j^k c_j^k,$$

and hence

$$S = S_N = \sum_{k=1}^{m_0+m_1} \sigma_k \geqq \sum_{k=1}^{m_0+m_1} \sum_{j=1}^{n} I_j^k c_j^k = \sum_{j=1}^{n} \sum_{k=1}^{m_0+m_1} I_j^k c_j^k = \sum_{j=1}^{n} \int_{-\infty}^{\infty} f_j(x)\, dx,$$

which proves inequality (4) if $\Delta=0$.

Next we consider the case $\Delta>0$.

Let $(x_1^0, x_2^0, \ldots, x_n^0)$ denote such a point where $H(f_1(x_1), f_2(x_2), \ldots, f_n(x_n))$ takes its maximum value and x_i^0 is the right-hand side end point of one of constant intervals of $f_i(x)$ on the axis x_i. The fact that $\sum_{i=1}^{n} x_i$ can be chosen from an interval $(t, t+\Delta)$ can be considered so that one of the intervals $[x_i^0-1, x_i^0]$ ($i=1, 2, \ldots, n$) is enlarged, e.g. for $i=1$, to $[x_1^0-1, x_1^0+\Delta]$ and on this enlarged interval we set $f_1(x_1)=f_1(x_1^0)$, furthermore everything is shifted by Δ to the right on $[x_1^0, \infty)$; and we estimate a similar integral as before. If we take the integral

$$\int_{x_1^0}^{x_1^0+\Delta} H(f_1(x_1), f_2(x_2^0), \ldots, f_n(x_n^0))\, dx_1,$$

this is obviously equal to

$$\Delta \cdot \max_{x_1, \ldots, x_n} H(f_1(x_1), f_2(x_2), \ldots, f_n(x_n));$$

and the rest of the integral

$$\int_{-\infty}^{\infty} \sup_{t \le \sum_{i=1}^{n} x_i \le t+\Delta} H(f_1(x_1), f_2(x_2), \ldots, f_n(x_n))\, dt$$

is not less than

$$\int_{-\infty}^{\infty} \sup_{\sum_{i=1}^{n} x_i = t} H(f_1(x_1), f_2(x_2), \ldots f_n(x_n))\, dt.$$

Hence and from the result proved in the case $\Delta=0$, (4) follows for $\Delta>0$, too. The proof is thus completed.

The Corollary can be deduced easily, we have just to note that considering the series as step functions $f_i(x_i)$, the left-hand side of (5) is not less than the left-hand side of (4) with $\Delta=l$.

References

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