# Finite type representations of infinite symmetric groups

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The group $G(S)$ of all permutations of a set $S$ of infinite cardinality $d$ is a topological group in the topology of pointwise convergence on $S$. The author has shown [3] that any continous unitary representation of $G(S)$ is the direct sum of irreducible representations, each (non-trivial) of which acts on a Hilbert space of dimension $d$. The author, on account of Theorem 1 below, conjectures that any unitary representation of $G(S)$ on a Hilbert space of dimension $d$ is continuous (and thus any unitary representation on a Hilbert space of dimension less than $d$ is trivial). Our results and conjecture seem analogous to certain theorems for Lie groups [2 and 5].

A representation of a group is of finite type if the von Neumann algebra generated by the range of the representation is of finite type [1, definition 5, p. 97].

Theorem. *Let $S$ be an infinite set and let $G(S)$ be the group of all permutations of $S$. Any non-trivial unitary representation of $G(S)$ of finite type acts on a Hilbert space of dimension greater than the cardinal of $S$. In particular, the permutation group of the integers has no non-trivial unitary representation of finite type on separable Hilbert space.*

Proof. Let $S_1$, $S_2$, and $S_3$ be pairwise disjoint subsets of $S$, with cardinal $(S_i)=$ $=$cardinal $(S)=d$ for $i=1, 2, 3$. Let $\varphi$ be a 1—1 correspondence between $S_1$ and $S_2$. If $s\in S_1$, let $p(s)$ be the permutation which interchanges $s$ with $\varphi(s)$ and leaves all other members of $S$ fixed. Let $Z$ be the set of all subsets of $S_1$ which have cardinality $d$. If $Y\in Z$ let $p(Y)=\pi_{s\in Y}p(s)$. Define an equivalence relation $\sim$ on $Z$ by $Y_1\sim Y_2$ iff the cardinal of the symmetric difference $Y_1 \Delta Y_2$ is less than $d$. Let $T$ be a subset of $Z$ which contains exactly one member of each equivalence class of $Z$ under $\sim$; then cardinal$(T)=2^d$. If $t_1$, $t_2\in T$, then $p(t_1)$ is contained in no proper normal subgroup of $G$ [4, p. 306], $p(t_1)=p(t_1)^{-1}$, $p(t_1)\,p(t_2)=p(t_1\Delta t_2)$, and $p(t_1)$ and $p(t_1\Delta t_2)$ are conjugate to each other since each of these permutations is the product of $d$ 2-cycles and also leaves $S_3$ elementwise fixed.

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Let $U$ be a unitary representation of $G$ on a Hilbert space $H$ such that the von Neumann algebra $U(G)''$ generated by $U(G)$ is of finite type [1, definition 5, p. 97]. Assume that no subrepresentation of $U$ is trivial. There is an ultraweakly continuous trace tr on $U(G)''$ such that $\mathrm{tr}(I_H)=1$, where $I_H$ is the identity operator on $H$. Without loss of generality, we may assume that there is a vector $v \varepsilon H$, $\|v\|=1$, such that $tr(W)=$ $=(Wv, v)$ for all $W \in U(G)''$. (If not, replace $U(g)$ by the representation $g \to U(g) \otimes I_K$, where $K$ is an infinite dimensional separable Hilbert space and $U(g) \otimes I_K$ acts on $H \otimes K$. [See 1, Theorem 1, p. 51].) We may assume that tr is faithful. (If not, replace $U$ by the restriction of $U$ to the closed subspace spanned by $U(G)v$.)

Let $V$ be the representation of $U(G)''$ determined by the vector state $W \to (Wv, v)$ of $U(G)''$ by the Gelfand—Segal construction. If $g \in G$, let $V_1(g)=V(U(g))$. Then $V_1$ is a representation of $G$ and is unitarily equivalent to the subrepresentation of $U$ on the closure of $U(G)''v$. $V_1$ acts on the Hilbert space which is the completion of $U(G)''$ with respect to the inner product $\langle W_1, W_2 \rangle = tr(W_2 * W_1)$.

If $g \in G$, then $\langle U(g), U(g) \rangle = tr(U(g) * U(g)) = tr(I_H) = 1$. If $t_1, t_2 \in T$ with $t_1 \neq t_2$, then $\langle U(p(t_1)), U(p(t_2)) \rangle = tr(U(p(t_2))^* U(p(t_1))) = tr(U(p(t_2)) U(P(t_1))) =$ $= tr(U(p(t_2) p(t_1))) = tr(U(p(t_1 \Delta t_2)))$. Let $tr(U(p(t_1))) = \alpha$. We have $\alpha \neq 1$ since for $g \in G$, the equality $tr(U(g))=1$ is equivalent to $U(g)=I_H$, and thus $\alpha=1$ would imply the triviality of $U$, because $p(t_1)$ is contained in no proper normal subgroup of $G$.

Since $p(t_1)$ is conjugate to $p(t_1 \Delta t_2)$, we have $tr(U(p(t_1 \Delta t_2)))=\alpha$. A simple computation shows that $\|U(p(t_1)) - U(p(t_2))\| = (2(1-\alpha))^{1/2}$; the norm on the Hilbert space on which $V_1$ acts. Consequently, the open balls of radius $((1-\alpha)/2)^{1/2}$ centered at the $U(p(t))$ with $t \in T$ are mutually disjoint open sets in the Hilbert space on which $V_1$ acts. Consequently, the dimension of this Hilbert space, and therefore the dimension of $H$, is at least $\text{G}^d$.

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