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A forbidden substructure characterization of Gauss codes

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GAUSS [2; pp. 272, 282—286] considered the following problem. Given a closed curve in the plane which is *normal*, i.e., lies in general position. Label the crossing points of the curve. The *Gauss code* of the curve is the word obtained by proceeding along the curve and noting each crossing point label as it is traversed. In the resulting word, every label occurs exactly twice. The problem is to characterize those words which are Gauss codes. Such words will be called here *realizable*. For a brief history of the work on the problem see [3; pp. 71—73]. In that reference, GRÜNBAUM says, "Solutions of the characterization problem have been found recently (TREYBIG [6], MARX [4]); however, they are of the same aesthetically rather unsatisfactory character as MacLane's criterion for the planarity of graphs. A characterization of Gauss codes in the spirit of the Kuratowski criterion for planarity of graphs is still missing." This work is an attempt to supply the "missing" criterion. The reader must be the judge of the aesthetic merits. Note that our characterization does meet EDMONDS' criterion [1] for a "good characterization".

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In the sequel, the symbols that make up a word are called *letters* and are denoted by capital Roman letters. Words and sequences of consecutive letters within a word are denoted by lower case Greek letters. For our purposes, two cyclic rearrangements of a word are equivalent. Given a word α, $|\alpha|$ is the number of letters in α.

Lemma 1. *Let A, A', B, B' be non-vertices on a normal planar curve G. Suppose no pair of edges (different from those containing A, A', B, B') separates A from A' and B from B'. Suppose G can be imbedded so that A and A' are on the boundary of the same face; similarly assume that G can be imbedded with B and B' on the same face.*

Then G has an imbedding with A and A' on the same face boundary and B and B' on the same face boundary. Also, the directions of the curve at A and A', relative to each other, are the same as in the hypothesized imbedding; similarly for B and B'.

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Proof. We use induction on the number of vertices.

I. Assume no two vertices of G can be separated by two edges. Then it is well known (see [5]) that the imbedding of G is essentially unique in the sense that the boundaries of faces are uniquely determined. Thus the assertion is trivial.

II. Assume there is a pair e, f of edges separating two vertices. There are three cases to consider.

IIa. Suppose e, f separate no two of A, A', B, B'. Let G_1 be the component of $G-e-f$ containing A, A', B, B' and G_2 the other. Let us replace G_2, e, f by an edge j connecting the endpoints of e and f in G_1. This way a new normal planar curve G', containing A, A', B, B', arises. Moreover, G' has imbeddings with A and A' (B and B') on one face. Hence by induction, G' has an imbedding with A and A' on the same face and B and B' on the same face. Clearly we can replace j by $e \cup G_2 \cup f$, obtaining an imbedding of G with the same property; further, the directions are as desired.

IIb. Suppose e, f separate e.g. A from A', B, B'. Let G_1 be the component of $G-e-f$ containing A', B, B' and G_2 the other. Replace G_2, e, f by an arc j connecting the endpoints of e and f in G_1 as above to obtain a curve G'. Select a point A'' on j. Then the hypothesized imbeddings of G yield an imbedding of G' with A' and A'' on the same face and another one with B and B' on the same face. It is easy to see that no pair of edges separates A' from A'' and B from B'. Therefore, G' has an imbedding with A' and A'' on the same face and, with B and B' on the same face, by the induction hvpothesis. In this imbedding A must be on the boundary of one of the two faces adjacent to e.f. Thus, replacing j by $e \cup G_2 \cup f$ (and "flipping over" G_2 if necessary) we obtain the desired imbedding of G. The directions are again easily seen to be as desired.

IIc. Assume e, f separate two of A, A', B, B' from the other two. By the assumption, they must separate $\{A, A'\}$ from $\{B, B'\}$. Then we can imbed the component of $G-e-f$ containing A and A' as in the hypothesized imbedding with A and A' on one face. We can imbed the other component as in the other hypothesized imbedding, and thus obtain the required imbedding of G.

Definition.

(1) For a word $\omega = A\alpha A\beta$ we define the *vertex split at A* to be the word $\omega_A = \alpha^{-1}\beta$.

(2) For a word $\omega = A\alpha A\beta$ we define the *loop removal at A* to be the word obtained by deleting A and both occurrences of the letters in α.

(3) A *subword* of a word ω is any word obtained by a sequence of vertex splits and loop removals.

(4) A word ω has the *parity condition* if between the two occurrences of any letter there are an even number of letters.

(5) A word ω has the *biparity condition* if given any unlinked vertices A, B with $\omega = A\alpha A\mu B\beta B\gamma$, α and β have an even number of letters in common.

The parity and biparity conditions are independent, necessary for planarity, but not sufficient (e.g. consider the word ABCDEFBADCFE).

Lemma 2. *Suppose* $\omega = A\alpha A\mu B\beta B\gamma$ *has* ω_A *and* ω_B *realizable and* α *and* β *have an even number of letters in common. If we cannot factor* $\alpha = \alpha_1\alpha_2$ *and* $\beta = \beta_1\beta_2$ *(the factors are assumed non-empty) so that* α_1, β_2, μ *have no letters in common with* α_2, B_1, γ, *then* ω *is realizable.*

Proof. Realize $A\alpha A\mu\beta^{-1}\gamma$ ($\alpha^{-1}\mu B\beta B\gamma$, respectively) and then split B (A, respectively). We get two realizations of $A\alpha^{-1}A'\mu B\beta^{-1}B'\gamma$, one with A and A' on the same face, one with B and B' on the same face. If some pair of edges separated A from A' and B from B', we could get a factorization such as we have ruled out. Thus, Lemma 1 applies and there is a realization with A and A' on the same face boundary; also B and B'. The directions are also proper for reconnection.

Let Γ be an arc from A to A' and Δ an arc from B to B', each spanning the face of which the points in question are boundary points. We may assume Δ and Γ intersect in at most one point. We show they are disjoint. Let Γ_1 be the arc from A to A' corresponding to α^{-1}; similarly Δ_1. Then $\Gamma \cup \Gamma_1$ and $\Delta \cup \Delta_1$ are closed curves intersecting in an even number of points. The intersections correspond to the common letters in α and β — even in number — and the intersections of Γ_1 and Δ_1. Thus $\Gamma_1 \cap \Delta_1 = \varphi$. Reconnect A and A' along Γ_1 and B and B' along Δ_1, giving a realization of ω.

Corollary. *Suppose* $\omega = A\alpha A\mu B\beta B\gamma$ *is not realizable but* ω_A *and* ω_B *are. Also* α *and* β *have an even number of letters in common. The following are ruled out*

(1) $\omega = AX\text{—}A\text{—}BX\text{—}B\text{—}$,

(2) $\omega = A\text{—}A\text{—}X\text{—}B\text{—}B\text{—}X\text{—}$,

(3) $\omega = AX\text{—}A\text{—}B\text{—}B\text{—}X\text{—}$,

(4) $\omega = AX\text{—}Y\text{—}Z\text{—}A\text{—}BY\text{—}X\text{—}Z\text{—}B\text{—}$.

Definition. We say ω is *critical* if it fails to be realizable, but its every vertex split is realizable.

Lemma 3. *Suppose* $\omega = A\alpha B\beta A\gamma B\delta$ *is critical, has biparity, and all letters of* α *and* γ *are contained in* δ. *Then* α *is empty if and only if* γ *is empty.*

Proof. Suppose e.g. $\alpha = \varnothing$ and $\gamma \neq \varnothing$. We have $\omega = AB\beta A\gamma_1 XB\delta_1 X\delta_2$. Apply Lemma 2 (1) to A and X, and obtain a contradiction. The case $\gamma = \varnothing$ is similar.

Lemma 4. *Suppose* $\omega = A\alpha B\beta A\gamma B\delta$ *is critical and satisfies the parity and biparity conditions and all letters of* α *and* γ *are contained in* δ. *Then* α *and* γ *are both empty.*

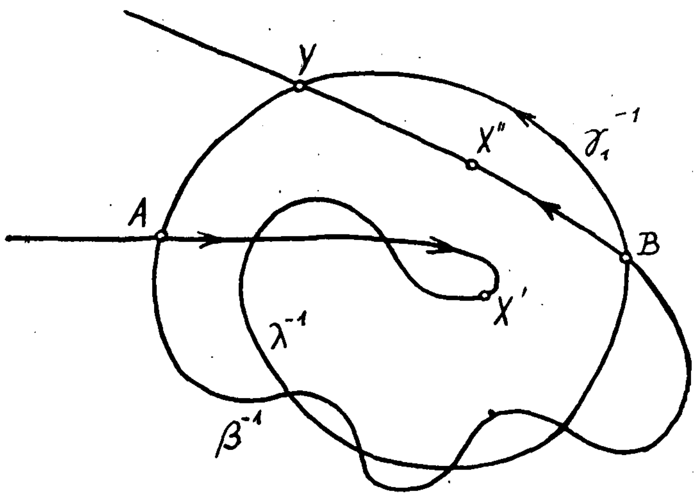
Proof. We may assume from Lemma 3 that both α and γ are non-empty. We shall obtain the contradiction that ω is realizable.

Let X and Y be two adjacent letters of α: by Corollary (2) of Lemma 2 the common letters of α and δ occur in the same order. Hence we can write $\omega = A\alpha_1XY\alpha_2B\beta A\gamma B\delta_1X\delta_2Y\delta_3$. We show $\delta_2=\emptyset$. To this end note that δ_2 has no letters in common with α_1 or δ_3, again by Corollary (2). Thus, Lemma 3 applies to ω written in the form $X\delta_2Y\delta_3A\alpha_1XY\alpha_2B\beta A\gamma B\delta_1$, and so $\delta_2=\emptyset$. We conclude $\omega = A\alpha B\beta A\gamma B\varepsilon_1\alpha\varepsilon_2$. Further, by Corollary (3) to Lemma 2, γ and ε_1 have no letters in common.

By a similar argument, or by applying the results of the preceding paragraph to ω^{-1}, we can write $\omega = A\alpha B\beta A\gamma B\lambda\alpha\zeta\gamma\mu$. From Corollary (2) we know that every letter of ζ occurs in β. Then Corollary (4) applies, where X is the first letter of α, B plays the role of Y, Z is a letter in ζ and β, and the last letter of γ plays the role of B; hence $\zeta=\emptyset$. Let $\alpha=\alpha_1X$, $\gamma=Y\gamma_1$. We write $\omega=A\alpha_1XB\beta AY\gamma_1B\lambda\alpha_1XY\gamma_1\mu$.

From the parity conditions on A, B, X, and Y, we can deduce that α_1 has even length. From the biparity condition on A and Y we deduce that β and λ have an even number of letters in common.

Now realize $\omega_X=A\alpha_1X'\alpha_1^{-1}\lambda^{-1}B\gamma_1^{-1}YA\beta^{-1}BX''Y\gamma_1\mu$. Because of the above parity arguments the curve $\alpha_1X'\alpha_1^{-1}$ and the arc $X''Y$ are inside the A loop and the directions are proper for reconnection (see figure).



We can then reconnect X' and X'', getting a relatization of ω. This contradiction completes the proof of the lemma.

Theorem. *Suppose ω is critical and satisfies the biparity condition. Then $\omega = A_1A_2\dots A_nA_1A_2\dots A_n$; the A_j are distinct, and n is even.*

Proof. First we show that either ω is in the desired form or it satisfies the parity condition. To this end note that for linking vertices A, X, parity on one implies parity on the other. Write $A\alpha X\beta A\gamma X\delta$. Split X and obtain $A\alpha\gamma^{-1}A\beta^{-1}\delta$ realizable; whence (i) $|\alpha|\equiv|\gamma|$. The parity condition for A is (ii) $|\alpha|\equiv|\beta|+1$; for X, (iii) $|\beta|+1\equiv|\gamma|$. Now (i) and (ii) are equivalent to (i) and (iii). Let A be an arbitrary vertex of ω. If any vertex B fails to link A, split it, and the parity for A is immediate. So consider the case where every vertex links A, and, even more in view of the preceding, every vertex must link every vertex that links A. The only such words have the form $A_1 \dots A_n A_1 \dots A_n$. If n is odd, ω is realizable; so we conclude n is even.

Now consider a critical ω with parity and biparity. We show this must lead to a contradiction. First, we recognize that ω must have at least two unlinked vertices; otherwise it has the form $A_1 \dots A_n A_1 \dots A_n$, n odd, and is realizable. We can select two such vertices so that $\omega = A\alpha AB\beta B\gamma$. Next we establish that α and β have no common letters. Let X be the first letter of β also in α; we can write $\omega = A\alpha_1 X\alpha_2 AB\beta_1 X\beta_2 B\gamma$. By choice of X, α_1 and α_2 have no letters in common with β_1; by Corollary (2) to Lemma 2, α_1 and α_2 are disjoint. Thus Lemma 4 applies to A and X, and we get the contradiction that $B\beta_1$ is empty. Finally, we can write $\omega = AY\alpha_1 AB\beta B\gamma_1 Y\gamma_2$, but this is ruled out by Corollary (3) of Lemma 2.

Theorem. *A word ω is realizable if and only if it contains no subword of the form $A_1 A_2 \dots A_n A_1 A_2 \dots A_n$, n even.*

Proof. If ω is realizable, it is easy to show it has no subword of the above form.

So suppose ω is not realizable. We proceed by induction on the number of vertices in ω. If this number is 2, $\omega = A_1 A_2 A_1 A_2$, the desired conclusion. So suppose the theorem true for all words of $<N$ vertices and let ω have N vertices.

By the induction hypothesis, we can assume ω is critical. If ω has biparity, apply the previous theorem, and the conclusion follows. If ω does not have biparity, then $\omega = A\alpha A\beta C\gamma C\delta$, where α and γ have an odd number of points in common. The realizable vertex split ω_A tells us that γ is even. From this, we see that the loop removal of A leaves us with a word without parity. Again apply the induction hypothesis.

References

[1] J. Edmonds, Minimum partition of a matroid into independent subsets, *J. Res. Nat. Bureau of Standards*, Sect. B., **69** (1965), 67—72.

[2] C. F. Gauss, *Werke*, Teubner (Leipzig, 1900).

[3] B. Grünbaum, *Arrangements and Spreads*, Regional Conference Series in Mathematics No. 10, American Mathematical Society, 1972.

[4] M. L. Marx, The Gauss Realizability Problem, *Proc. Amer. Math. Soc.*, **22** (1969), 610—613.

[5] J. v. Sz.-Nagy, Über ein topologisches Problem von Gauss, *Math. Z.*, **26** (1927), 579—592.

[6] L. B. Treybig, A Characterization of the Double Point Structure of the Projection of a Polygonal Knot in Regular Position, *Trans. Amer. Math. Soc.*, **130** (1968), 223—247.

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