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# An inequality for functions

MASAMI OKADA and KÔZÔ YABUTA

**1.** The main purpose of this note is to prove the following:

Theorem 1. *Let* $(X, \Sigma, m)$ *be a probability measure space and* $0<p<q<\infty$. *Let* $f\in L^q(m)$, $\int |f|^q dm=1$ *and* $A_0^p=\int |f|^p dm$. *Suppose* $0<A<A_0$ *and let* $c>0$, $y>1$ *satisfy the equation*

$$\frac{1}{c}=\frac{y^q-A^q}{1-A^q}=\frac{y^p-A^p}{A_0^p-A^p}. \tag{1}$$

*Then*

$$m\{x\in X; |f(x)|>A\}\geqq c.$$

*Equality holds if and only if there exists a measurable set* $S$ *with* $m(S)=1-c$ *and* $|f|=A$ *on* $S$ *and* $|f|=y$ *on* $X\setminus S$.

This result shows that the above constant $c$ is the best possible for the function class $\{f\in L^q(m); \|f\|_q=1$ and $\|f\|_p=A_0\}$ and is a refinement of an inequality given in Burkholder and Gundy [1, *p.* 258, Lemma 2. 3]. Applications of inequalities of this type are also found in Zygmund [3, p. 216—p. 217]. Also this is a generalization of an inequality for analytic functions in Kamowitz [2, p. 236, Theorem B]. His result follows from the next theorem, which is an immediate corollary of his Lemma 3 in the case of non-atomic measure space, and which also in the general setting can be proved in the same way as in the proof of Theorem 1.

Theorem 2. *Let* $(X, \Sigma, m)$ *be a probability measure space and* $0<p<\infty$. *Let* $f\in L^p(m)$, $\int |f|^p dm=1$ *and* $\log |f|\in L^1(m)$, $A_0=\exp\int \log |f| dm$. *Suppose* $0<A<A_0$ *and let* $c>0$, $y>1$ *satisfy the equation*

$$\frac{1}{c}=\frac{y^p-A^p}{1-A^p}=\frac{\log y-\log A}{\log A_0-\log A}. \tag{2}$$

*Then*

$$m\{x\in X; |f(x)|>A\}\geqq c.$$

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*Equality holds if and only if there exists a measurable set $S$ with $m(S)=1-c$ and $|f|=A$ on $S$ and $|f|=y$ on $X\setminus S$.*

As for Kamowitz's results we shall discuss them in the last section. To prove Theorems 1 and 2 we have improved the method of the proof of Kamowitz's theorem in [2]. We shall prove in the next section Theorem 1 only and omit the proof of Theorem 2.

**2.** We state first three elementary lemmas.

Lemma 1. *Given $0<A<A_0<1$, $0<p<1$, there exist unique $c$ and $y$ such that $0<c<1$, $y>1$ and*

$$1=(1-c)A+cy \quad \text{and} \quad A_0^p=(1-c)A^p+cy^p. \tag{3}$$

*Further $y$ satisfies the equation*

$$\frac{y-A}{1-A}=\frac{y^p-A^p}{A_0^p-A^p}. \tag{4}$$

*Also, for fixed $A$, the solution $y$ of (4) ($y>1$) decreases when $A_0$ increases and, for fixed $A_0$, it increases when $A$ increases.*

Proof. It is obvious that the equation (4) has a unique solution for $y>1$, since $y=A$ is a solution of (4) and $1<(1-A^p)(A_0^p-A^p)^{-1}$. Let $c=(1-A)(y-A)^{-1}$, for this $y$. Then $c$ and $y$ satisfy equation (3) and $0<c<1$. Also by elementary calculation one sees that the last assertions hold.

Lemma 2. *Let $(X, \Sigma, m)$ be a finite positive measure space. If $0<p<1$, $0<A\leqq 1$, $G\in L^\infty(m)$ and $|G|\leqq A$, then*

$$p\Big(m(X)A-\int_X |G|\,dm\Big)\leqq m(X)A^p-\int_X |G|^p\,dm.$$

*Equality holds only when $|G|=A$.*

Proof. By elementary computation one has the inequality $p(A-t)\leqq A^p-t^p$ for $0<t\leqq A$. Integrating the both sides of the inequality $p(A-|G|)\leqq A^p-|G|^p$, we have the desired one. It is then clear that the equality holds only when $|G|=A$.

Lemma 3. *Let $0<A<1$, $0<p<1$, $\beta\geqq 1$, $y>1$ and $0\leqq pa\leqq b$. Then*

$$b+\beta y^p-(\beta-1)A^p\geqq \big(a+\beta y-(\beta-1)A\big)^p.$$

*Equality holds if and only if $\beta=1$ and $a=b=0$.*

Proof. Note that

$$b+\beta y^p-(\beta-1)A^p = b+y^p+(\beta-1)(y^p-A^p),$$
$$a+\beta y-(\beta-1)A = a+y+(\beta-1)(y-A).$$

Let

$$g(y) = \{(y^p+b+(\beta-1)(y^p-A^p))^{1/p}-(y+a+(\beta-1)(y-A))\}/y.$$

Then we get

$$g(y) = \{1+s+(\beta-1)(1-B^p)\}^{1/p}-\{1+t+(\beta-1)(1-B)\},$$

where $t=a/y$, $s=b/y^p$ and $B=A/y$. Now we have clearly

$$g(y) \geqq 1+(s+(\beta-1)(1-B^p))/p-(1+t+(\beta-1)(1-B)) =$$
$$(s-pt)/p+(\beta-1)(1-B^p-p(1-B))/p.$$

Further by the assumption $b\geqq ap$ we have

$$s-pt = (by^{1-p}-ap)/y \geqq (b-ap)/y \geqq 0,$$

and as in Lemma 2 we see $1-B^p\geqq p(1-B)$. Hence we have $g(y)\geqq 0$. It is then obvious that $g(y)=0$ if and only if $\beta=1$ and $ap=b=0$.

**3.** Now we are in the position to prove Theorem 1 for $q=1$.

Proof. Assume $d=m\{|f|>A\}<c$. Let $S=\{|f|\leqq A\}$ and $S'=X\setminus S$. Then we have by Lemma 1

$$\int_S |f|+\int_{S'} |f| = 1 = (1-c)A+cy = (1-d)A+cy-(c-d)A, \tag{5}$$
$$\int_S |f|^p+\int_{S'} |f|^p = A_0^p = (1-c)A^p+cy^p = (1-d)A^p+cy^p-(c-d)A^p.$$

By Hölder's inequality one gets

$$\int_{S'} \frac{|f|^p}{d}\, dm \leqq \Big(\int_{S'} \frac{|f|}{d}\, dm\Big)^p. \tag{6}$$

Combining this with (5) we have

$$\frac{(1-d)A^p-\int_S |f|^p}{d}+\frac{c}{d}y^p-\left(\frac{c}{d}-1\right)A^p \leqq \left(\frac{(1-d)A-\int_S |f|}{d}+\frac{c}{d}y-\left(\frac{c}{d}-1\right)A\right)^p. \tag{7}$$

However by Lemmas 2 and 3 we have the converse inequality and hence the equality, which implies $c=d$, a contradiction. Next suppose the equality holds in (6) and let $S=\{|f|\leqq A\}$ and $S'=\{|f|>A\}$. Then we have $m(S)=1-c$ and we see by the above argument that the equality holds in (7), which implies $\int_S |f|dm=(1-c)\,A$ and that the equality holds in (6). Hence we get $|f|=A$ on $S$ and $|f|$ is constant on $S'$. This value is $y$ by (5). The proof is complete.

**4.** Now Theorem 1 follows immediately from the special case above. In fact, let $g=|f|^q$ in the setting of Theorem 1. Then

$$\int g\,dm = 1, \quad \int g^{p/q}dm = \int |f|^p dm = (A_0^q)^{p/q} \quad \text{and} \quad \{|f| > A\} = \{g > A^q\}.$$

Theorem 1 results if we replace the $y$ of the case $q=1$, by $y^q$.

**5. Application.** Let $f(z)$ be an analytic function in the open unit disc in the complex plane which lies in the Hardy space $H^p$ for some $0<p<\infty$, i.e., let

$$\|f\|_p = \sup_{0<r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta\right)^{1/p} < \infty, \quad \text{and} \quad F(\theta) = \lim_{r\to 1} f(re^{i\theta}).$$

Then $\log|F(\theta)|$ is integrable unless $F(\theta)\equiv 0$ and one has by Jensen's inequality for $H^p$ functions

$$\log|f(0)| \leqq \frac{1}{2\pi}\int_0^{2\pi}\log|F(\theta)|\,d\theta.$$

One sees easily that the constant $c$ in Theorem 2 is an increasing function of $A_0$ for fixed $A$. Hence applying Theorem 2 we have the following theorem of Kamowitz.

Theorem 3. *Let* $f\in H^p$, $0<p<\infty$ *and* $\|f\|_{H^p}=1$. *If* $0<A<|f(0)|$, *then* $m\{0\leqq\theta\leqq 2\pi;\ |F(\theta)|>A\}\geqq c$, *where* $c=(1-A^p)(y^p-A^p)^{-1}$ *and* $y$ *is determined by the equation* (2) *in Theorem* 2 *for* $A_0=|f(0)|$. *This constant is the best possible. Here* $m$ *denotes the normalized Lebesgue measure on* $[0, 2\pi]$.

That the constant c is the best possible is shown by the $H^\infty$ outer function defined by

$$\exp\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log g(\theta)\,d\theta,$$

where $g(\theta)=A$ for $0<\theta\leqq 2\pi(1-c)$ and $=y$ for $2\pi(1-c)<\theta\leqq 2\pi$.

One can also formulate Theorem 1 for $H^p$ functions, and also in this case i. is shown by the above function that the arising constant $c$ is the best possiblet Finally we remark that Kamowitz uses the inner-outer factorization theorem for $H^p$ functions and he states Theorem 3 only for $1\leqq p<\infty$.

## References

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MATH. INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, JAPAN