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An inclusion theorem for normal operators

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1. Introduction

In the present remark we present a simple result relating, for normal operators, the spectrum of a submatrix to the spectrum of the whole matrix.

First some terminological conventions. Given a linear operator A on a Hilbert space H and a disc $D=\{z: |z-z_0|\leqq r\}$ in the complex plane we shall say that D is an inclusion disc for A if $D\cap\sigma(A)$ is nonvoid; here, of course, $\sigma(A)$ stands for the spectrum of A. Given an orthogonal projection P in H (a bounded linear operator P such that $P^2=P$ and $P^*=P$) with range H_0 we shall denote by A_P the restriction to H_0 of the operator PAP (or PA, which is the same).

We shall denote by Q the operator $I-P$; Q is the orthogonal projection whose range is $H_0^\perp$.

We shall prove two inclusion theorems, one for "matrices" and one for operators.

Although the main idea of the proof is the same, we prefer to treat the finite-dimensional and infinite-dimensional cases separately. First of all, the proof becomes simpler in finite-dimensional spaces since in this case the spectrum coincides with the point spectrum. Second, owing to a regrettable lack of communication between functional analysts and specialists in finite-dimensional problems the result could easily be overlooked if presented as a corollary of a result in functional analysis.

2. The inclusion theorems

We begin with the finite-dimensional case.

(2.1) Theorem. *Let A be a normal operator on a finite-dimensional Hilbert space H. Let P be an orthogonal projection in H. If $\lambda_P\in\sigma(A_P)$ then there exists a $\lambda\in\sigma(A)$ such that $|\lambda-\lambda_P|\leqq|QAP|$.*

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Proof. Since $\lambda_P \in \sigma(A_P)$ there exists a vector x such that $|x|=1$, $x=Px$ and $(PA-\lambda_P)x=0$. We have then

$$(A-\lambda_P)x = (P+Q)(A-\lambda_P)x = Q(A-\lambda_P)x = Q(A-\lambda_P)Px = QAPx,$$
$$((A-\lambda_P)^*(A-\lambda_P)x, x) = |(A-\lambda_P)x|^2 = |QAPx|^2 \leqq |QAP|^2.$$

Let ξ be the minimum of the quadratic form corresponding to $(A-\lambda_P)^*(A-\lambda_P)$ on the unit sphere of H. It follows that $\xi \leqq |QAP|^2$. Since ξ belongs to the spectrum of $(A-\lambda_P)^*(A-\lambda_P)$ and $A-\lambda_P$ is normal, there exists a proper value λ of A such that $|\lambda-\lambda_P|^2=\xi$ whence $|\lambda-\lambda_P| \leqq |QAP|$. The proof is complete.

To extend this result to the infinite-dimensional case small changes have to be made in the statement and in the proof. We need the notion of the approximate point spectrum $\sigma_a(T)$ of a linear operator T on a Banach space E. We say that λ belongs to the approximate point spectrum of T if $\inf\{|(T-\lambda)x|;\ x\in E,\ |x|=1\}=0$. Clearly $\sigma_a(T)\subset\sigma(T)$; if $\lambda\in\sigma_a(T)$, the equation

$$(T-\lambda)x = 0$$

need not have nontrivial solutions but does have approximate solutions: for each $\varepsilon>0$ there exists a vector x of norm one such that $|(T-\lambda)x|<\varepsilon$. Now we may state the inclusion theorem.

(2.2) Theorem. *Let A be a normal operator on a Hilbert space H. Let P be an orthogonal projection in H and set $Q=I-P$. Then each disc of diameter $|QAP|$ and centre in $\sigma_a(A_P)$ intersects the spectrum of A.*

Proof. Suppose that $\lambda\in\sigma_a(A_P)$ and that $x=Px$. We have then

$$(A-\lambda)x = (P+Q)(A-\lambda)Px = P(AP-\lambda)x+Q(AP-\lambda)Px = P(AP-\lambda)x+QAPx$$

and, since P and Q are projections on H_0 and $H_0^\perp$

$$|(A-\lambda)x|^2 = |P(AP-\lambda)x|^2+|QAPx|^2 = |(A_P-\lambda)x|^2+|QAPx|^2.$$

Since $\lambda\in\sigma_a(A_P)$, it follows that the infimum of $|(A-\lambda)x|^2$ on the unit sphere of H_0 is $\leqq|QAP|^2$. Consequently, the infimum of $|(A-\lambda)x|$ on the unit sphere of H is $\leqq|QAP|$. It follows that the disc $|z|\leqq|QAP|$ must contain a point of $\sigma(A-\lambda)$ so that the disc $|z-\lambda|\leqq|QAP|$ must contain a point of $\sigma(A)$. The proof is complete.

Let us add a few remarks concerning applications of the preceding theorem. In order to obtain inclusion discs for the operator A, we must have some information about the approximate point spectrum of the "smaller" operator $T=PAP$ restricted to the range of P. It is a well known fact that, in general, the approximate point spectrum $\sigma_a(T)$ of an operator T, although always nonempty, may differ considerably from the whole spectrum. If T is normal, we have $\sigma_a(T)=\sigma(T)$. However, the restriction of PAP to the range of P need not be normal if A is normal; for the restriction of PAP to be normal it suffices that P commute with A. If A happens to be symmetric then T is symmetric as well. The only other case likely to be of use is that of a finite-dimensional projection P; then T is finite-dimensional so that $\sigma_a(T)=\sigma(T)$.

Using these remarks, we may now state a corollary of the theorem with approximate point spectrum replaced by the whole spectrum.

(2.3) *Let A be a normal operator on a Hilbert space H. Let P be an orthogonal projection in H. Denote by A_P the operator PAP restricted to the range of P. Then each disc of diameter $|(I-P)AP|$ and centre in $\sigma(A_P)$ is an inclusion disc for A provided one of the following conditions is satisfied:*

1° A_P is normal, 2° $PA=AP$, 3° A is symmetric, 4° P is finite-dimensional.

3. Some consequences

In this section we formulate three immediate consequences of the theorem in important particular cases.

First we investigate one-dimensional projections. Clearly each such projection is given by the formula $Px=(x, e)e$ where e is an arbitrary vector of norm one.

(3.1) *Let A be a normal operator on a Hilbert space H. Let e be a vector of norm one. Then the disc*

$$|z-(Ae, e)| \leqq \big|(A-(Ae, e))e\big|$$

contains at least one point of the spectrum of A.

Proof. Clearly,

$$PAPx = (x, e)(Ae, e)e, \quad QAPx = APx-PAPx = (x, e)\big(A-(Ae, e)\big)e,$$

and A_P has a one-point spectrum (Ae, e). The conclusion follows immediately from the theorem.

Another particular case of interest is that of projections onto a hyperplane.

(3.2) *Let A be a normal operator on a Hilbert space H. Let e be a vector of norm one. Let P be defined by*

$$Px = x-(x, e)e.$$

Then each disc of diameter $|(A-(Ae, e))e|$ and centre in $\sigma_a(A_P)$ intersects the spectrum of A.

Proof. It suffices to compute QAP. We have for every $x\in H$

$$QAPx = (APx, e)e = \big(A(x-(x, e)e), e\big)e =$$
$$= \big((Ax, e)-(x, e)(Ae, e)\big)e = \big(x, A^*e-(Ae, e)^*e\big)e.$$

It follows that

$$|QAP| = \left|(A-(Ae,e))^* e\right| = \left|(A-(Ae,e))e\right|.$$

The last equality is a consequence of the fact that, for a normal operator A, the operator $A-(Ae, e)$ is normal as well.

To conclude we present a result formulated in the classical language of "matrix theory".

(3.3) Theorem. *Let A be a complex n by n matrix with elements a_{ik}. Denote by $A^{(i)}$ the $n-1$ by $n-1$ matrix obtained by deleting the* ith *row and* ith *column of A. Suppose that A is normal so that the following equality holds*

$$\sum_{\substack{k=1\\k\neq i}}^{n} |a_{ik}|^2 = \sum_{\substack{j=1\\j\neq i}}^{n} |a_{ji}|^2 \text{ for each } i.$$

Denote by r_i the nonnegative square root of this number. Then:

1° *Each disc of the form $|z-a_{ii}|\leqq r_i$ contains at least one proper value of A.*

2° *Each disc of the form $|z-\alpha|\leqq r_i$, where α is a proper value of $A^{(i)}$, contains at least one proper value of A.*

Proof. An immediate consequence of the preceding two results.

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