# Uniformly distributed sequences in quotient groups

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Let $G$ be a compact topological group with countable base, $H$ a closed normal subgroup, $p: G \to G/H$ the canonical homomorphism. If a sequence $(x_n)$ is uniformly distributed in $G$, then it is easy to prove that $p(x_n)$ is u.d. in $G/H$. If $G=K\times H$, and $(y_n)$ is, u.d. in $K$, then as is proved in [1], for almost every sequence $(z_n)$, $z_n\in H$ (with respect to the product-measure on $H^\infty$) the sequence $(y_n, z_n)$ is u.d. in $G$. We prove the following

Theorem 1. *If $(y_n)$ is u.d. in $G/H$, then there exists a u.d. sequence $(x_n)$ in $G$ such that $p(x_n)=y_n$.*

Remark. The result in [1] is based on a Theorem of HLAWKA ([3], Th. 11) using a theorem of Hill on infinite matrices. Here we are going to use a different method.

The main result of this paper is the following

Proposition. *Let $G$ be a locally compact group, $H$ a closed normal amenable subgroup such that $G/H$ is compact If $(y_n)$ is u.d. in $G/H=K$, then for any $f\in L^1(G)$, $\int f(x)\,dx=0$ ($dx$ = left Haar measure on $G$) there exists a sequence $(x_n)$ in $G$, satisfying $p(x_n)=y_n$ and $\lim_{N\to\infty}\left\|\frac{1}{N}\sum_{n=1}^{N} x_n f\right\|_1=0$ ($\|g\|_1=\int|g(x)|\,dx$ and $yf(x)=f(y^{-1}x)$).*

Theorem 1 then follows from the following

Lemma 1. *Let $G$ be a compact metric group, then there exists an $f\in L^1(G)$ such that $\int f(x)\,dx=0$ and $\frac{1}{N}\left\|\sum_{n=1}^{N} x_n f\right\|_1\to 0$ implies: $(x_n)$ is u.d. in $G$.*

Proof. We may choose an $f\in L^2(G)$ such that $\int f(x)\,dx=0$ and $\int f(x)D(x)\,dx$ is a non-singular matrix for any non-trivial continuous irreducible unitary representation $D$ of $G$ (there are only countably many inequivalent ones) and then apply Th. 2 of [6].

If $G$ is compact so is $G/H$ and $H$, $H$ is amenable ([4], Ch. 8) and Theorem 1 follows from the Proposition and Lemma 1.

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## Proof of the Proposition

Lemma 2. *Given $\varepsilon > 0$, there exists a sequence $(x_n)$ such that* (i) $p(x_n) = y_n$ *and* (ii) $\limsup_{N\to\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} x_n f \right\|_1 \leqq \varepsilon$.

Proof. Let $T_H$: $L^1(G) \to L^1(G/H)$, $[T_H f](p(x)) = \int_H f(xy)\,dy$ be the canonical morphism onto $L^1(K)$, $dy$= left Haar-measure on $H$, and put $g = T_H f$, then $\int_K g(x)\,dx = 0$ by Weil's formula ([4], Ch. 3, §4. 4, 5). Choose a neighbourhood of $U$ of the neutral element of $G$ such that

(1) $\quad \|xf - f\|_1 < \varepsilon$ for all $x \in U$, put $V = p(U)$ ([4], §5.5).

There exist finitely many elements $b_1, \ldots, b_m$ in $K = G/H$ such that

$$\bigcup_{i=1}^{m} b_i V = K. \quad \text{Put} \quad B_l = b_l V - \bigcup_{i=1}^{l-1} b_i V \quad (l = 1, \ldots, m).$$

Then $B_1, \ldots, B_m$ constitute a partition of $K$ into measurable sets. Let $\chi_i$ denote the characteristic function of $B_i$. Then we have

(2) $$\sum_{i=1}^{m} \chi_i * g = 1 * g = \int g(x)\,dx = 0.$$

If $v \in V$, choose $u \in U$ such that $p(u) = v$, then by means of the relation $T_H(uf) = vT_H f$ and by (1) we obtain

$$\|(b_i v)g - b_i g\|_1 = \|vg - g\|_1 = \|T_H(uf - f)\|_1 \leqq \|uf - f\|_1 \leqq \varepsilon,$$

thus we have

(3) $$\left\| \chi_i * g - \left(\int \chi_i\right) b_i g \right\|_1 \leqq \int_{b_i V} \chi_i(y) \|yg - g\|_1\,dy < \varepsilon \left(\int \chi_i\right).$$

Choose elements $a_1, \ldots, a_m$ from $G$ in such a way that $p(a_i) = b_i$ and set $f_1 = \sum_{i=1}^{m} \left(\int \chi_i\right) a_i f$. Then we have $\|T_H f_1\|_1 < \varepsilon$ ((2)+(3)). We have assumed that $H$ was amenable, therefore there exist elements $s_1, \ldots, s_r \in H$ such that

(4) $$\frac{1}{r} \left\| \sum_{k=1}^{r} s_k f_1 \right\|_1 < \varepsilon \quad \text{([4], Ch. 8, §4.3, §6.5).}$$

We may suppose that the boundary of $V$ has measure 0. (If not, replace $V$ by a neighbourhood $V'$ of the neutral element of $K$ that is contained in $V$ and whose boundary has measure 0, also replace $U$ by $U \cap p^{-1}(V)$). Then we have

(5) $$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_i(y_n) = \int_K \chi_i(x)\,dx \quad \text{([2], Th. 13).}$$

For $i=1, \ldots, m$ and $j=1, 2, \ldots$ let $n(j, i)$ be that index $n$ of $y$ for which $y_n \in B_i$ and exactly $j$ members of the sequence $y_1, \ldots, y_n$ belong to $B_i$. Then we have

(6)
$$y_{n(j,i)} = b_i v_{n(j,i)}, \quad v_{n(j,i)} \in V.$$

Define the sequence $(z_n)$ in $G$ by $z_{n(j,i)} = s_k a_i$ if $j \equiv (k-1) \bmod r$, then by (4) and (5) we obtain:

(7)
$$\limsup_{N\to\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} z_n f \right\|_1 \leqq \varepsilon.$$

Choose finally $u_n \in U$ suh that $p(u_n) = v_n$, $x_n = z_n u_n$, then $\|z_n f - x_n f\|_1 = \|u_n f - f\|_1$ and by (1) we obtain that $\limsup_{N\to\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} x_n f \right\|_1 \leqq 2\varepsilon$. This completes the proof of Lemma 2. Now let $x_{n,k}$ be the sequence obtained by Lemma 2 for $\varepsilon = 1/2k$, then we can find a strictly increasing sequence of positive integers $N_k$ satisfying

(8)
$$\text{a)} \quad \left\| \frac{1}{N} \sum_{n=1}^{N} x_{n,k} f \right\|_1 \leqq \frac{1}{k}, \quad N \geqq N_k, \quad \text{b)} \quad N_1 + \ldots + N_k \leqq N_{k+1}.$$

We define: $x_n = x_{n,k+1}$ if $N_k < n \leqq N_{k+1}$; $k = 0, 1, \ldots, N_0 = 0$, then (8) a) implies that $\left\| \sum_{n=M+1}^{N} x_{n,k} f \right\|_1 \leqq (N+M)/k$, $N > M \geqq N_k$, thus by (8) b) we obtain that for $N_k < N \leqq N_{k+1}$ we have

$$\left\| \sum_{n=1}^{N} x_n f \right\|_1 \leqq N_1 \|f\|_1 + (N_1 + N_2) + (N_2 + N_3)/2 + \ldots + (N_{k-1} + N_k)/(k-1) + $$
$$+ (N_k + N)/k = o(N)$$

and the proof of the Proposition is complete.

As a further application of the Proposition we obtain

Theorem 2. *If $(y_n)$ is a uniformly distributed sequence modulo* 1, *then there exists a sequence $(x_n)$ such that $x_n \equiv y_n$ (mod 1) and $(x_n)$ is u.d. modulo $a$ for all $a > 0$.*

Proof. We apply the Proposition to $f \in L^1(R)$ satisfying $\hat{f}(t) \neq 0$ iff $t \neq 0$, then there exists a sequence $(x_n)$ such that $p(x_n) = y_n$ and $\lim_N \left\| \frac{1}{N} \sum_{n=1}^{N} x_n f \right\|_1 = 0$, and by direct computation we obtain that $\lim \frac{1}{N} \sum_{n=1}^{N} \exp(iyx_n) = 0$ for all $y \neq 0$, which proves Theorem 2 by means of Weyl's criterion.

Example. If $z$ is an arbitrary irrational number then the sequence $(nz)$ is u.d. mod 1, therefore there exists a sequence $(x_n)$ congruent to $(nz)$ mod 1 and such that $(x_n)$ is u.d. mod $a$ for all $a > 0$, whereas $(nz)$ is u.d. mod $a$ iff $a$ is an irrational multiple of $z$.

Remarks. A stronger version of the Proposition is true: there exists a single sequence that satisfies the relation in the Proposition for all $f \in L^1(G)$, $\int f = 0$ (compare [7], Th. 1, Th. 2). This gives a partial answer to a question in [5], (starting from a countable dense set of $L^0(G) = \{f\colon f \in L^1(G). \int f = 0\}$ a similar proof leads to this result. $G$ must be second countable.) Theorem 1 remains valid if $G$ is compact and $H$ has a countable dense subset. It can be shown that there exists a sequence $(s_n)$ such that $\lim \left\|\frac{1}{r}\sum_{n=1}^{r} s_n f\right\|_1 = \|T_H f\|_1$ for all $f \in L^1(G)$ (construction and proof as in [8]). The same proof as that of the Proposition (compare (4)!) shows that there exists a sequence $(x_n)$, $p(x_n) = y_n$ such that $\lim \left\|\frac{1}{N}\sum_{n=1}^{N} x_n f\right\|_1 = 0$ for all $f \in L^1(G)$, $\int f = 0$, which implies that $(x_n)$ is u.d. in $G$ ([7], Th. 2).

Finally, it should be noted that the condition that $H$ is amenable in the assumptions of the Proposition is necessary ([4], Ch. 8, § 4.3).

Additional Remark (by proof-reading). Th. 1 implies immediately: *Let $G$, $G_1$ be compact metric groups, $p\colon G \Rightarrow G_1$ a continuous homorphism. If $(y_n)$ is u. d. in $p(G)$, then there exist $x_n$, $p(x_n) = y_n$, $(x_n)$ is u. d. in $G$.*

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