Acta Sci. Math., 38 (1976), 171—182

Idempotent reducts of abelian groups

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1. Introduction. The aim of this paper is to describe all idempotent reducts of abelian groups, in particular all minimal nontrivial idempotent reducts and to characterize the lattice of all subclones of the clone of the full idempotent reduct of abelian groups. These results extend a theorem of PŁONKA (see [2], [3]) which states that the clones of the idempotent reducts of a (not necessarily abelian) group form a chain if and only if the group is abelian and of prime power exponent. Moreover, if an abelian group is of exponent p^k for a prime p ($k \in N$) then this chain consists of $k+1$ elements. Our main result (Theorem 1) gives a representation for any idempotent reduct of the group of integers as a finite intersection of reducts of a very simple type. Hence the further results mentioned above can be deduced easily.

Basic universal algebraic concepts are from [1]. We are only interested in algebras up to equivalence. Let $\langle A; P\rangle$ be an algebra where P can be supposed to be the set of all polynomials. Reducts of $\langle A; P\rangle$ are defined to be algebras of the form $\langle A; R\rangle$ with $R \subseteq P$. By an idempotent reduct of $\langle A; P\rangle$ we mean a reduct $\langle A; J\rangle$ with all operations in J idempotent. The maximal idempotent reduct of $\langle A; P\rangle$, i.e. the reduct $\langle A; I\rangle$ where I contains all the idempotent operations of P, will be called the full idempotent reduct.

We adopt the definition of a *clone* due to TAYLOR. In [5] a clone is defined to be a heterogeneous algebra $\langle A_k;\ C_m^n,\ e_i^n\rangle_{k,m,n,i \in N, i \leqq n}$ with heterogeneous operations

$$C_m^n : A_n \times A_m^n \to A_m$$

called *substitutions* and

$$e_i^n : \{\emptyset\} \to A_n$$

called *projections*, satisfying the identities:

$$C_m^p(z, C_m^n(y_1, x_1, \ldots, x_n), \ldots, C_m^n(y_p, x_1, \ldots, x_n)) =$$
$$= C_m^n(C_n^p(z, y_1, \ldots, y_p), x_1, \ldots, x_n), \quad n, m, p \in N = \{1, 2, \ldots\};$$
$$C_m^n(e_i^n, x_1, \ldots, x_n) = x_i, \quad m, n, i \in N, \quad i \leqq n;$$
$$C_n^n(y, e_1^n, \ldots, e_n^n) = y, \quad n \in N.$$

Received June 28, 1975.

The concepts of isomorphism, subalgebra, subalgebra generated by a subset, etc. can naturally be generalized for heterogeneous algebras, in particular for clones, too (see [5]).

Note that for any algebra $\langle A; P\rangle$ the set of all polynomials P is a clone and the reducts of $\langle A, P\rangle$ are determined up to equivalence by the subclones of P.

The following notations will be used in the paper. N_0 or N will stand for the set of nonnegative or positive integers, respectively. Z and Z_m will mean the set of integers and the set of integers modulo m ($m\in N_0$), respectively. The greatest common divisor of natural numbers m and n will be denoted by (m, n). If e is an element of a lattice L we shall write $[e)_L$ for the dual principal ideal of L generated by e. The subclone of a clone C generated by the subset H of C will be denoted by $[H]$.

Any n-ary ($n\in N$) polynomial of an abelian group $\langle G; +,-, 0\rangle$ is of the form $\langle g_1, \dots, g_n\rangle\mapsto c_1g_1+\dots+c_ng_n$ where $c_1, \dots, c_n\in Z$. It will be denoted by $c_1x_1+\dots$ $\dots+c_nx_n|_G$. Such a polynomial is idempotent if $c_1+\dots+c_n=1$. In particular, $c_1x_1+\dots+c_nx_n|_Z$ is idempotent if and only if $c_1+\dots+c_n=1$.

2. The main theorem. Let n be a natural number. Consider the set of all idempotent polynomials $c_1x_1+\dots+c_mx_m|_Z$ with the property that all the coefficients c_i but one are divisible by n. Obviously, they form a clone for which we shall write $Cl(n)$. In particular, the clone of the full idempotent reduct of $\langle Z; +, -, 0\rangle$ coincides with $Cl(1)$, while $Cl(0)$ is the clone consisting of all the projections only. Note that $Cl(n)$ consists exactly of those polynomials $c_1x_1+\dots+c_mx_m|_Z$ for which $c_1x_1+\dots$ $\dots+c_mx_m|_{Z_n}$ is a projection.

Theorem 1. *For any clone C with $Cl(1)\supset C\supset Cl(0)$ there exist uniquely determined pairwise relatively prime numbers $p_1, \dots, p_k>1$ such that*

$$C = \bigcap\big(Cl(p_i)|1 \leq i \leq k\big). \tag{1}$$

We prepare the proof of the theorem by stating several lemmas. For simplicity subscript Z in polynomials will be omitted.

Lemma 1. *If $(Cl(1)\supseteq)C\ni x+(-n)y+nz$ ($n\in N_0$) then C together with any polynomial $c_1x_1+\dots+c_mx_m$ contains each polynomial $(c_1+t_1n)x_1+\dots+(c_m+t_mn)x_m$ with $t_1, \dots, t_m\in Z$ and $t_1+\dots+t_m=0$. In particular, $Cl(n)$ is generated by the polynomial $x+(-n)y+nz$ and, consequently,*

$$[Cl(m)\cup Cl(n)] = Cl\big((m, n)\big), \quad m, n\in N_0. \tag{2}$$

Proof. First we prove our claim for $C=[\{x+(-n)y+nz\}]$, i.e. we prove $[\{x+(-n)y+nz\}]=Cl(n)$. Inclusion $\subseteq$ is obvious. Inclusion in the opposite direction follows in two steps. By induction on t we get

$$x+(-tn)y+tnz\in C,$$

then by induction on r we can prove that for any $d_1x_1+\ldots+d_rx_r \in Cl(n)$ and $i \neq j$

$$d_1x_1+\ldots+d_rx_r = \left(\sum_{\substack{k=1 \\ k\neq i}}^{r} d_kx_k+d_ix_j\right)+(-d_i)x_j+d_ix_i \in C,$$

as required.

Let C stand now for an arbitrary clone containing the polynomial $x+(-n)y+nz$. Obviously $C \supseteq Cl(n)$; hence if $c_1x_1+\ldots+c_mx_m \in C$ and $t_1, \ldots, t_m \in Z$ with $t_1+\ldots+t_m=0$ then

$$(c_1+t_1n)x_1+\ldots+(c_m+t_mn)x_m =$$

$$= (c_1x_1+\ldots+c_mx_m)+t_1nx_1+\ldots+t_mnx_m \in C$$

which was to be proved.

As for (2) we note that

$$x+(-(m,n))y+(m,n)z = (x+(-um)y+umz)+(-vn)y+vnz$$

where $u, v \in Z$ and $um+vn=(m,n)$. This implies inclusion $\supseteq$ in (2). Inclusion $\subseteq$ is obvious, thus the proof of the lemma is complete.

Lemma 2. *If* $(Cl(1)\supseteq)C\supset Cl(p)$, *where p is a prime, then* $C=Cl(1)$.

Proof. If C is properly contained in $Cl(1)$ then the polynomials $c_1x_1+\ldots+c_mx_m|_{Z_p}$ where $c_1x_1+\ldots+c_mx_m \in C$ constitute a proper subclone in the clone of the full idempotent reduct of $\langle Z_p; +, -, 0\rangle$. This contradicts the theorem of Płonka quoted in the introduction.

Lemma 3. *Let* $n \in N$, $n \geqq 2$ *and let* $p_1, \ldots, p_n > 1$ *be pairwise relatively prime numbers. If*

$$(Cl(1) \supseteq)C \supset Cl(p_1p_n) \cap (\cap (Cl(p_j)| 2 \leqq j \leqq n-1))$$

and C contains a polynomial

$$d_1x_1+\ldots+d_mx_m \in \cap (Cl(p_j)| 1 \leqq j \leqq n)$$

such that there exist two coefficients in $\langle d_1, \ldots, d_m\rangle$ *not divisible by* p_1p_n *(for brevity we will say that this polynomial separates* p_1 *and* p_n*) then*

$$C \supseteq \cap (Cl(p_j)| 1 \leqq j \leqq n).$$

Proof. Set $p=p_1\ldots p_n$. By Lemma 1, we can assume $p \nmid d_i$, $i=1, \ldots, m$. Moreover, we can suppose $d_i=e_iq_i$, where

$$q_i = \frac{p}{p_{j_{i-1}+1}\cdots p_{j_i}} \quad (i = 1, \ldots, m),$$

and $0=j_0<j_1<\ldots<j_m=n$. First we show that

$$\frac{p}{p_1}u_1x_1+\ldots+\frac{p}{p_n}u_nx_n\in C \tag{3}$$

whenever $\sum_{i=1}^{n}\frac{p}{p_i}u_i=1$. This is obvious for $n=2$. Suppose $n\geqq 3$. Let

$$f_iq_i=\sum_{j=j_{i-1}+1}^{j_i}\frac{p}{p_j}u_j \quad (i=1,\ldots,m).$$

We have $\sum_{i=1}^{m}e_iq_i=\sum_{i=1}^{m}f_iq_i=1$ and $(q_1,\ldots,q_{i-1},q_{i+1},\ldots,q_m)=\frac{p}{q_i}$, thus $p|(e_i-f_i)q_i$, $i=1,\ldots,m$. As $Cl(p)\subseteqq C$ we can apply Lemma 1 to have

$$f_1q_1x_1+\ldots+f_mq_mx_m\in C.$$

Choose integers v_1, v_2 such that $v_1p_1p_n+v_2p_2\ldots p_{n-1}=1$. Clearly

$$v_1p_1p_nx+v_2p_2\ldots p_{n-1}y\in C,$$

$$\left(\frac{p}{p_1}u_1+\frac{p}{p_n}u_n\right)x_1+\frac{p}{p_2}u_2x_2+\ldots+\frac{p}{p_{n-1}}u_{n-1}x_{n-1}\in C,$$

thus

$$v_2p_2\ldots p_{n-1}(f_1q_1x_1+(1-f_1q_1)x_n)+$$

$$+v_1p_1p_n\left(\left(\frac{p}{p_1}u_1+\frac{p}{p_n}u_n\right)x_1+\frac{p}{p_2}u_2x_2+\ldots+\frac{p}{p_{n-1}}u_{n-1}x_{n-1}\right)=$$

$$=\left(v_2p_2\ldots p_{n-1}\left(\frac{p}{p_1}u_1+\ldots+\frac{p}{p_{j_1}}u_{j_1}\right)+v_1p_1p_n\left(\frac{p}{p_1}u_1+\frac{p}{p_n}u_n\right)\right)x_1+$$

$$+(1-v_2p_2\ldots p_{n-1})\frac{p}{p_2}u_2x_2+\ldots+(1-v_2p_2\ldots p_{n-1})\frac{p}{p_{n-1}}u_{n-1}x_{n-1}+$$

$$+\left(v_2p_2\ldots p_{n-1}\left(\frac{p}{p_{j_1+1}}u_{j_1+1}+\ldots+\frac{p}{p_{n-1}}u_{n-1}\right)+(1-v_1p_1p_n)\frac{p}{p_n}u_n\right)x_n=$$

$$=\left(\frac{p}{p_1}u_1+t_1p\right)x_1+\ldots+\left(\frac{p}{p_n}u_n+t_np\right)x_n\in C$$

where $t_1,\ldots,t_n$ are integers with $t_1+\ldots+t_n=0$. This implies (3) by Lemma 1.

Finally we drop the assumption $n\geqq 3$ and prove that

$$\left[Cl(p)\cup\left\{\frac{p}{p_1}u_1x_1+\ldots+\frac{p}{p_n}u_nx_n\right\}\right]=\bigcap(Cl(p_i)|1\leqq i\leqq n). \tag{4}$$

Let we denote the clone on the left by D. Suppose

$$d_1'x_1+\ldots+d_m'x_m\in\bigcap(Cl(p_i)|1\leqq i\leqq n).$$

Using the above notations we can suppose $d_i'=e_iq_i$ $(i=1, \ldots, m)$, because (4) is symmetric in $p_1, \ldots, p_n$. Applying $p|(e_i-f_i)q_i$ and

$$f_1q_1x_1+\ldots+f_mq_mx_m\in D$$

we have

$$d_1'x_1+\ldots+d_m'x_m\in D$$

proving that $D\supseteq\bigcap(Cl(p_i)|1\leqq i\leqq n)$. Inclusion $\subseteq$ is obvious. The proof of the lemma is complete.

Lemma 4. *Let $m\in N$ and let $q_1, \ldots, q_m, q>0$ be pairwise relatively prime numbers. If*

$$(5)\qquad (Cl(1)\supseteq)C\supset Cl(q_1^{k_1}\ldots q_m^{k_m}q)$$

and C contains the polynomial

$$vq_1^{j_1}\ldots q_m^{j_m}qx+(1-vq_1^{j_1}\ldots q_m^{j_m}q)y,$$

where $v\in Z$, $(v, q_1\ldots q_m)=1$ and $1\leqq j_i<k_i$ $(i=1, \ldots m)$, then

$$C\supseteq Cl(q_1^{j_1}\ldots q_m^{j_m}q).$$

Proof. Let us introduce the notations

$$p=q_1^{j_1}\ldots q_m^{j_m}q,\quad p'=q_1^{k_1}\ldots q_m^{k_m}q,\quad t=\frac{p'}{p}.$$

First suppose $p'|p^2$. By induction on r we show that

$$P_r(x_0, \ldots, x_r)=(1-rvp)x_0+vpx_1+\ldots+vpx_r\in C.$$

$P_1(x_0, x_1)$ is the polynomial given above, and for $r\geqq 2$ we have

$$P_r(x_0, \ldots, x_r)=((1-vp)P_{r-1}(x_0, \ldots, x_{r-1})+vpx_r)+$$
$$+(1-r)v^2p^2x_0+v^2p^2x_1+\ldots+v^2p^2x_{r-1},$$

where $p'|p^2$. Thus

$$x_0+vpx_1+(-vp)x_2=P_t(x_0, x_1, x_2, \ldots, x_2)+vp'x_0+(-vp')x_2\in C.$$

Applying Lemma 1 and $(vp, p')=p$, we have $C\supseteq Cl(p)$, as was to be proved.

By the assumption of the lemma there exists a natural number k such that $p'|p^{2^k}$. We can choose k to be minimal with this property. We prove the lemma by induction on k. For $k=1$, the statement was proved in the preceding paragraph. Suppose $k\geqq 2$ and the lemma is true for $k-1$. Obviously

$$v^2p^2x+(1-v^2p^2)y=vp(vpx+(1-vp)y)+(1-vp)y\in C,$$

$(v^2, q_1\ldots q_m)=1$ and $p'|(p^2)^{2^{k-1}}$, which implies

$$(5')\qquad C\supseteq Cl(q_1^{2j_1}\ldots q_m^{2j_m}q).$$

We can now apply the lemma (case $k=1$) for (5) substituted by (5′), hence we have

$$C \supseteq Cl(q_1^{j_1} \dots q_m^{j_m} q),$$

completing the proof of the lemma.

Lemma 5. *Let $p>2$, $q\geqq 1$ be relatively prime numbers. If*

$$(Cl(1) \supseteq) C \supset Cl(pq)$$

and C contains a polynomial

$$vqx+(1-vq)y$$

with $v\in Z$, $(v, p)=(1-vq, p)=1$, then $C\supseteq Cl(q)$.

Proof. Let φ denote Euler's function. Using congruences

$$(1-vq)^{\varphi(p)} \equiv 1 \pmod p$$

implied by $(1-vq, p)=1$ and

$$(1-vq)^{\varphi(p)} \equiv 1 \pmod q,$$

we have

$$(1-vq)^{\varphi(p)} = 1+v'pq, \quad v'\in Z.$$

Clearly

$$(1-vq)^{\varphi(p)-1}\big((1-vq)x+vqy\big)+\big(1-(1-vq)^{\varphi(p)-1}\big)z =$$

$$= (1+v'pq)x+(1-vq)^{\varphi(p)-1}vqy+\big(1-(1-vq)^{\varphi(p)-1}\big)z\in C$$

therefore by Lemma 1 and $Cl(pq)\subseteq C$ we have

$$x+uqy+(-uq)z\in C,$$

where $u=(1-vq)^{\varphi(p)-1}v$ and $(u, p)=1$. Applying again Lemma 1 we conclude $C\supseteq Cl(q)$, which completes the proof of the lemma.

Lemma 6. *Let $p_1, p_2, p_3\geqq 1$ be pairwise relatively prime odd numbers. If*

$$(Cl(1) \supseteq) C \supset Cl(2p_1p_2p_3)$$

and C contains a polynomial

$$v_1p_2p_3x_1+v_2p_1p_3x_2+v_3p_1p_2x_3,$$

where v_i, $i=1, 2, 3$ are odd integers, then $C\supseteq Cl(p_1p_2p_3)$.

Proof. We have

$$v_3p_1p_2(v_1p_2p_3x_1+v_2p_1p_3x_2+v_3p_1p_2x_3)+(1-v_3p_1p_2)x_3 =$$

$$= (v_1v_3p_2)p_1p_2p_3x_1+(v_2v_3p_1)p_1p_2p_3x_2+(1-2tp_1p_2p_3)x_3\in C$$

with $v_1v_3p_2$ and $v_2v_3p_1$ odd and $C \supseteqq Cl(2p_1p_2p_3)$, therefore by Lemma 1 we have

$$x_3 + up_1p_2p_3x_1 + (-up_1p_2p_3)x_2 \in C,$$

where $u = v_1v_3p_2$. We can apply again Lemma 1 to complete the proof.

We remark that Lemma 2 for $p>2$ is the special case $q=1$ of Lemma 5 and Lemma 2 for $p=2$ is the special case $p_1=p_2=p_3=1$ of Lemma 6.

Lemma 7. *For any clone C with* $Cl(1) \supseteqq C \supset Cl(0)$ *there exists a natural number* $n>0$ *such that* $C \supseteqq Cl(n)$.

Proof. By assumption C does not coincide with the trivial clone containing projections only and thus contains a polynomial $(1-k)x+ky$ for an integer $k \geqq 2$. If $k=2$ then

$$C \supseteqq [\{(-1)x+2y\}] = [\{(-1)(2x+(-1)y)+2z\}] = Cl(2).$$

Suppose now $k \geqq 3$. By induction on r it follows that

$$P_r(x, y) = rk^{r-1}(1-k)x + \big(1 - rk^{r-1}(1-k)\big)y \in C.$$

This is clear for $r=1$ and supposing to be true for r it is true also for $r+1$, because

$$P_{r+1}(x, y) = kP_r(x, y) + (1-k)\big(k^r x + (1-k^r)y\big)$$

and $k^r x + (1-k^r)y$ is obviously contained in C. Clearly $n = k^{k-2}(1-k)^2$ is even and

$$nx + (1-n)y = (1-k)^2\big(k^{k-2}x + (1-k^{k-2})y\big) + \big(1-(1-k)^2\big)y \in C,$$

$$(-n)x + (1+n)y = P_{k-1}(x, y) \in C.$$

To show the inclusion $C \supseteqq Cl(n)$ observe that

$$n\big(nx+(1-n)y\big) + (1-n)\big((1+n)x+(-n)z\big) = x + n(1-n)y + n(n-1)z \in C,$$

$$(-n)\big((-n)x+(1+n)z\big) + (1+n)\big((1-n)x+ny\big) = x + n(1+n)y + n(-n-1)z \in C,$$

and $\big(n(n-1), n(n+1)\big) = n$, which by Lemma 1 completes the proof of the lemma.

Proof of the theorem. By Lemma 7, there exists a natural number $n \geqq 1$ such that $C \supseteqq Cl(n)$. First we show the existence of $p_1, \dots, p_k$ in (1) under the assumption

$$C \subseteqq \bigcap \big(Cl(q_j^{t_j}) \,|\, 1 \leqq j \leqq m\big) \tag{6}$$

where $q_1, \dots, q_m$ are distinct primes and the prime factorization of n is $n = q_1^{t_1} \dots q_m^{t_m}$. To show (1) it suffices to prove the following statement: if $p_1, \dots, p_k > 1$ $(k \in N)$ are pairwise relatively prime numbers with $p_1 \dots p_k = n$ and

$$C \supset \bigcap \big(Cl(p_j) \,|\, 1 \leqq j \leqq k\big),$$

then there exists an $i \in N$ with $1 \leqq i \leqq k$ and integers $p_i', p_i'' > 1$ such that $(p_i', p_i'') = 1$, $p_i' p_i'' = p_i$ and

$$C \supseteqq \cap (Cl(p_j) \mid 1 \leqq j \leqq k, j \neq i) \cap Cl(p_i') \cap Cl(p_i'').$$

Suppose the conditions of this statement are satisfied by C and

$$d_1 x_1 + \ldots + d_r x_r \in C - \cap (Cl(p_j) \mid 1 \leqq j \leqq k) \quad (r \geqq 2).$$

This means that there exist two coefficients d_{i_1}, d_{i_2}, $1 \leqq i_1 < i_2 \leqq r$ and an index i, $1 \leqq i \leqq k$ such that $p_i \nmid d_{i_1}, d_{i_2}$. By symmetry we can assume $i_1 = 1$, $i_2 = 2$. Now (6) implies that for each $j (1 \leqq j \leqq m)$ all the coefficients but one in $\langle d_1, \ldots, d_r \rangle$ are divisible by $q_j^{t_j}$. Consequently, (p_i, d_1) or (p_i, d_2) is greater than 1, say $(p_i, d_1) = p_i'' > 1$. Moreover, if we set $p_i' = \dfrac{p_i}{p_i''}$ and $d_1' = \dfrac{d_1}{p_i''}$, then we obtain $p_i' | d_j$ for $j = 2, \ldots, r$, hence $p_i' | d_2 + \ldots + d_r = 1 - d_1$ and obviously $p_i' > 1$. Choose integers u, v such that $u \dfrac{n}{p_i} + v p_i = 1$. Since

$$u \frac{n}{p_i} (d_1' p_i'' x + (1 - d_1) y) + v p_i y = u d_1' \frac{n}{p_i'} x + \left((1 - d_1) u \frac{n}{p_i} + v p_i \right) y \in C$$

and this polynomial separates p_i' and p_i'', by Lemma 3 the proof of the statement is complete.

It has remained to prove that (6) holds if n is chosen to be minimal with the property $C \supseteqq Cl(n)$. Suppose, otherwise,

$$H = C - \cap (Cl(q_j^{t_j}) \mid 1 \leqq j \leqq m) \neq \emptyset.$$

First we show that H contains a binary polynomial. Assume that either all primes q_i $(i = 1, 2, \ldots, m)$ are odd or $4 | q_1^{t_1} \ldots q_m^{t_m}$. Let $d_1 x_1 + \ldots + d_t x_t \in H$, $t \geqq 2$. Then there are at least two coefficients not divisible by one of the prime powers $q_i^{t_i}$, say by $q_1^{t_1}$. Suppose all the coefficients not divisible by $q_1^{t_1}$ are $d_1, \ldots, d_r$ $(r \geqq 2)$ and $d_1, \ldots, d_s$ $(s \geqq 1)$ are not even divisible by q_1. If $s = 1$ then $q_1^{t_1} \nmid d_2$, $q_1 \nmid 1 - d_2$, hence $d_2 x + (1 - d_2) y \in H$. If $s > 1$ and, say, $d_1 \not\equiv 1 \pmod{q_1}$, then $q_1^{t_1} \nmid d_1$, $q_1 \nmid 1 - d_1$, thus $d_1 x + (1 - d_1) y \in H$, while if $d_i \equiv 1 \pmod{q_1}$ for all i, $1 \leqq i \leqq s$. then $s \geqq q_1 + 1 \geqq 3$, hence $d_1 + d_2 \equiv 2 \pmod{q_1}$ implies $(d_1 + d_2) x + (1 - d_1 - d_2) y \in H$.

We reduce the remaining case $q_1^{t_1} = 2$ to the one settled in the previous paragraph by proving that

$$(H \supseteqq) H' = C - \cap (Cl(q_j^{t_j}) | 2 \leqq j \leqq m) \neq \emptyset.$$

Suppose that, in contrary to our claim, $q_1^{t_1} = 2$ and $C \subseteqq \cap (Cl(q_j^{t_j}) | 2 \leqq j \leqq m)$. Then H contains a ternary polynomial which can naturally be supposed to have form

$$u_1 p_2 p_3 x + u_2 p_1 p_3 y + u_3 p_1 p_2 z$$

where u_i $(i=1, 2, 3)$ are odd integers and $p_i=q_{r_{i-1}+1}^{t_{r_{i-1}+1}}\dots q_{r_i}^{t_{r_i}}$ with $1=r_0\leqq r_1\leqq r_2\leqq r_3=m$. Applying Lemma 6 we have $C\supseteqq Cl(q_2^{t_2}\dots q_m^{t_m})$, contradicting the minimality of n.

Any binary polynomial in H can be written in the form

$$u_1q_1^{s_1}\dots q_r^{s_r}x+u_2q_{r+1}^{s_{r+1}}\dots q_k^{s_k}y \tag{7}$$

with $0\leqq r\leqq k\leqq m$, $(u_1, n)=(u_2, n)=1$ and $s_i\geqq 1$, $(i=1, 2, \dots, k)$. For brevity, we introduce the following notations:

$$p_1=q_1^{t_1}\dots q_r^{t_r},\quad p_2=q_{r+1}^{t_{r+1}}\dots q_k^{t_k},\quad p_3=q_{k+1}^{t_{k+1}}\dots q_m^{t_m}=\frac{n}{p_1p_2},$$

$$p_1'=q_1^{s_1}\dots q_r^{s_r},\quad p_2'=q_{r+1}^{s_{r+1}}\dots q_k^{s_k}.$$

(a) If both $r=k=0$, i.e. $p_1'=p_2'=1$, then applying Lemma 5 we get $C=Cl(1)$ which contradicts the minimality assumption. Hence we can suppose $r\geqq 1$.

(b) If $k=m$, then there exists an index i with $s_i<t_i$, $1\leqq i\leqq m$. Set $v_j=\min(s_j, t_j)$ $(j=1, 2, \dots, m)$. Since

$$u_1u_2p_1'p_2'x+(1-u_1u_2p_1'p_2')y\in C,$$

by Lemma 4 we have $C\supseteqq Cl(q_1^{v_1}\dots q_m^{v_m})$, contradicting the minimality of n.

(c) Let $k<m$, i.e. $p_3>1$. Clearly,

$$p_2'p_3\,|\,1-(u_1p_1')^{\varphi(p_3)}$$

for $u_1p_1'+u_2p_2'=1$, thus (b) applies to polynomial

$$(u_1p_1')^{\varphi(p_3)}x+\left(1-(u_1p_1')^{\varphi(p_3)}\right)y\in C,$$

provided the product of the two coefficients are not divisible by n. In the opposite case by Lemma 3 we have

$$C\supseteqq Cl(p_1)\cap Cl(p_2p_3). \tag{8}$$

If $s_i<t_i$ is satisfied for an i, $1\leqq i\leqq k$, say for $i=1$, then by (8) we can choose integers v_1, v_2 such that

$$v_1p_2p_3(u_1p_1'x+u_2p_2'y)+v_2p_1y\in C,$$

hence again applies (b). (In case $i>r$ the role of p_1 and p_2 has to be interchanged in (8), too.)

Finally, if $p_i|p_i'$ $(i=1, 2)$ then we rewrite the polynomial (7) in the form $u_1'p_1x+u_2'p_2y$, where $(u_1', p_2p_3)=1$, $(u_2', p_1p_3)=1$. Applying again (8) we have

$$v_2p_1(u_1'p_1x+u_2'p_2y)+v_1p_2p_3x\in C,$$

where $(v_2p_1u_1'p_1+v_1p_2p_3, p_3)=(v_2, p_3)=1$, $(v_2p_1u_2'p_2, p_3)=1$, thus obviously $p_3\neq 2$. Hence by Lemma 5 we have $C\supseteqq Cl(p_1p_2)$ which contradicts the minimality of n. The existence of $p_1, \dots, p_k$ in Theorem 1 is proved.

To prove uniqueness is suffices to show that for any two sequences of pairwise relatively prime numbers $p_1, \ldots, p_k$ ($k \geqq 1$) and $\bar{p}_1, \ldots, \bar{p}_m$ ($m \geqq 1$) the inclusion

$$\bigcap (Cl(p_i) \mid 1 \leqq i \leqq k) \subseteqq \bigcap (Cl(\bar{p}_j) \mid 1 \leqq j \leqq m)$$

implies that each $\bar{p}_j$ divides one of the p_i-s.

Let we denote the clones above by C and $\bar{C}$, respectively. Set $p = p_1 \ldots p_k$ and $\bar{p} = \bar{p}_1 \ldots \bar{p}_m$. Choose integers u_i, $1 \leqq i \leqq k$ such that $\sum_{i=1}^{k} \frac{p}{p_i} u_i = 1$. Since

$$\frac{p}{p_1} u_1 x_1 + \ldots + \frac{p}{p_k} u_k x_k \in C \subseteqq \bar{C}$$

all the coefficients but one of this polynomial are divisible by $\bar{p}_j$. Assume

$$\bar{p}_j \,\Big|\, \left(\frac{p}{p_2} u_2, \ldots, \frac{p}{p_k} u_k\right) = u p_1. \tag{9}$$

Applying the obvious inclusion $Cl(p) \subseteqq \bar{C}$ and the fact that $\bar{p}_1, \ldots, \bar{p}_m$ are pairwise relatively prime we have

$$\bar{p} \mid p. \tag{10}$$

Suppose $\bar{p}_j \nmid p_1$. Now (9) and (10) together with the equations $(p_1, p_i) = 1$, $(i = 2, \ldots, k)$ imply u to have a prime factor v with $v \nmid p_1$ and $v \mid \bar{p}_j$ and hence with $v \mid \frac{p}{p_1}$. Then

$$v \,\Big|\, \frac{p}{p_1} u_1 + \left(\frac{p}{p_2} u_2 + \ldots + \frac{p}{p_k} u_k\right) = 1.$$

This contradiction implies $\bar{p}_j \mid p_1$, hence the proof of Theorem 1 is complete.

3. Applications. Next we state two propositions that reduce the problem of describing all idempotent reducts of an arbitrary abelian group to that of the infinite cyclic group $\langle Z; +, -, 0\rangle$. Sometimes it will be convenient to use notation Z_0 instead of Z.

Proposition 1. *The clone of an abelian group $\langle G; +, -, 0\rangle$ is isomorphic to that of $\langle Z_i; +, -, 0\rangle$, where $i=0$ or $i=m(\geqq 1)$ according to whether $\langle G; +, -, 0\rangle$ satisfies no nontrivial identity or is of exponent m. In both cases the following map is an isomorphism:*

$$c_1 x_1 + \ldots + c_n x_n|_G \mapsto c_1 x_1 + \ldots + c_n x_n|_{Z_i}$$

for every $\langle c_1, \ldots, c_n\rangle \in Z_i^n$.

In particular, this map is an isomorphism between the clones of the full idempotent reducts of $\langle G; +, -, 0\rangle$ and $\langle Z_i; +, -, 0\rangle$.

Proposition 2. *For every $n \in N$ the lattice L_n of all subclones of the clone of the full idempotent reduct of $\langle Z_n; +, -, 0\rangle$ is isomorphic to the dual principal ideal $[Cl(n))_{L_0}$ of L_0 generated by $Cl(n)$, where L_0 is the lattice of all subclones of the clone of the full idempotent reduct of $\langle Z; +, -, 0\rangle$. The following map is an isomorphism:*

$$(Cl(n) \subseteqq)C \mapsto \{d_1x_1 + \ldots + d_m x_m|_{Z_n} \mid d_1x_1 + \ldots + d_m x_m|_Z \in C\}.$$

The proof of these propositions is straightforward and is therefore left to the reader.

Theorem 2. *Let $\langle G; I\rangle$ denote the full idempotent reduct of the abelian group $\langle G; +, -, 0\rangle$.*

(i) *If $\langle G; +, -, 0\rangle$ satisfies no nontrivial identity, then the lattice received from the lattice of all subclones of I by omitting the least element (i.e., the clone of projections) is dually isomorphic to the subdirect product of the partition lattice $E(N)$ of the set N and countably infinite samples of the chain $Q = \{0 \leqq 1 \leqq \ldots\}$ defined as follows:*

$$\mathscr{L}_0 = \{\langle \pi, s\rangle \mid \pi \in E(N),\ s \in Q^N, \text{ all but a finite number of components of } s \text{ equal } 0, \\ s(i) = 0 \text{ implies that } i \text{ constitutes in } \pi \text{ a class in itself}\}$$

(ii) *If $\langle G; +, -, 0\rangle$ is of exponent $n(>1)$ with prime factorization $n = p_1^{t_1} \ldots p_m^{t_m}$, then the lattice of subclones of the clone I is dually isomorphic to the subdirect product of the partition lattice $E(m)$ of the set $\{1, 2, \ldots, m\}$ and the chains $Q_i = \{0 \leqq 1 \leqq \ldots \leqq t_i\}$, $i = 1, 2, \ldots, m$ defined as follows:*

$$\mathscr{L}_m^{(t_1, \ldots, t_m)} = \{\langle \pi, s_1, \ldots, s_m\rangle \mid \pi \in E(m),\ s_i \in Q_i,\ s_i = 0 \text{ implies that } i \text{ constitutes in } \pi \\ \text{ a class in itself}\}$$

Proof. First we prove (i). By Proposition 1 it suffices to prove it for the group $\langle Z; +, -, 0\rangle$. By Lemma 7 the subset obtained from L_0 by omitting its least element constitutes a sublattice of L_0 which will be denoted by $\tilde{L}_0$. Consider the following map:

$$\psi : \mathscr{L}_0 \to \tilde{L}_0$$

$$\langle \pi, s\rangle \mapsto \bigcap (Cl(\prod_{i \in c} p_i^{s(i)}) \mid c \in \mathfrak{C}(\pi)) \tag{11}$$

where $\mathfrak{C}(\pi)$ means the set of classes corresponding to π and $\{p_1, \ldots, p_k, \ldots\}$ is the set of all primes.

By the definition of $\mathscr{L}_0$ all but a finite number of terms in the meet in (11) equal $Cl(1)$, thus ψ is a map into $\tilde{L}_0$. Obviously, ψ is a monotone order reversing map. Theorem 1 implies ψ to be onto, moreover, in the proof of Theorem 1 we showed uniqueness just by proving that ψ is invertible and ψ^{-1} is also monotonic. Hence ψ is an isomorphism which was to be proved.

By Propositions 1 and 2 (ii) is an easy consequence of (i).

Theorem 3. *An abelian group satisfying no nontrivial identity has no minimal nontrivial idempotent reduct.*

Let $n>1$ be a natural number with prime factorization $n=p_1^{t_1}...p_m^{t_m}$. The clones of the minimal nontrivial idempotent reducts of any abelian group $\langle G; +, -, 0\rangle$ of exponent n are the following ones:

(a) $[\{x+n_iy+(-n_i)z|_G\}]$ *for* $n_i=\frac{n}{p_i}$ $1\leqq i\leqq m$, *provided either* $p_i^2|n$ *or* $m=1$ *and* $n=p_1$;

(b) $[\{u_1q_1x+u_2q_2y|_G\}]$ *for all pairs of integers* $q_1, q_2>1$ *with* $(q_1, q_2)=1$, $q_1q_2=n$ *and* $u_1q_1+u_2q_2=1$ $(u_1, u_2\in Z)$.

Proof. Our first assertion is an obvious consequence of Theorem 2. To prove the second statement observe that by Theorem 2 and by Lemma 3 the clones covering $Cl(n)$ in L_0 (i.e. the clones $C\supset Cl(n)$ having the property that there does not exist any clone $C'\in L_0$ with $C\supset C'\supset Cl(n)$) are the following ones:

(a)′ $$Cl(n_i) = [\{x+n_iy+(-n_i)z|_Z\}],$$

(b)′ $$Cl(q_1)\cap Cl(q_2) = [Cl(n)\cup\{u_1q_1x+u_2q_2y|_Z\}].$$

We used the same notation as in (a) and (b). Now we can apply Propositions 1 and 2 to complete the proof of the theorem.

Remark. Since ψ given in the proof of Theorem 2 is an isomorphism, for any sequence $p_1, \dots, p_n$ of pairwise relatively prime numbers the interval $\left[Cl\left(\prod_{i=1}^{n} p_i\right), \bigcap\left(Cl(p_i)|1\leqq i\leqq n\right)\right]_{L_0}$ is dually isomorphic to the partition lattice $E(n)$. Hence we can apply the result of Sachs [4] to obtain that the lattice L_0 generates the variety of all lattices.

The author expresses her thanks to Dr. B. Csákány for his kind help in the preparation of this paper.

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