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A note on reductive operators

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A bounded linear operator T on a Hilbert space H is called a *reductive operator* if every invariant subspace of T reduces T. In this note, we shall study a local spectral theoretic condition which is satisfied by certain types of reductive operators. Consequently we shall obtain a set of conditions which are sufficient for a reductive operator to be equal to the sum of a normal operator and a commuting quasinilpotent operator. This will provide alternative proofs of some of the results of JAFARIAN [2], NORDGREN—RADJAVI—ROSENTHAL [3] and RADJABALIPOUR [4]. We shall be using the notation and terminology of DUNFORD and SCHWARTZ [1].

If an operator T has the single valued extension property, called property (A), then it satisfies Dunford's condition (B) if $\sigma(T, x)\cap\sigma(T, y)=\varphi$ implies that $\|x\|\leqq\leqq K\|x+y\|$, where K is independent of x and y. In [5], STAMPFLI introduced an orthogonality version of condition (B), that is, $\sigma(T, x)\cap\sigma(T, y)=\varphi$ implies that $(x, y)=0$, for all vectors x and y in the Hilbert space H. An operator T with property (A) satisfies Dunford's condition (C) if for each closed set δ, $H(\delta)=\{x\in H: \sigma(T, x)\subset\delta\}$ is closed subspace. A basic theorem of DUNFORD [1, page 2147] asserts that an operator, T on a Hilbert space H is spectral (i.e., $T=S+Q$ where S is similar to a normal operator, Q is quasinilpotent and $SQ=QS$) if and only if T satisfies conditions (A), (B), (C), and (D). It is easy to prove that if in this result the condition (B) is replaced by the orthogonality version of condition (B) then S will be a normal operator and conversely [5, Lemma 7].

Proposition 1. *If T is a reductive operator and if T satisfies* (A) *and* (C) *then T satisfies the orthogonality version of condition* (B).

Proof. For any closed set δ, since T is reductive, $TP_\delta=P_\delta T$ where P_δ denotes the projection of H onto $H(\delta)$. Thus $\sigma(T, P_\delta x)\subset\sigma(T, x)$ for all $x\in H$. If $\sigma(T, x)\cap\cap\sigma(T, y)=\varphi$ then $\sigma(T, P_\delta y)\subset\sigma(T, y)\cap\delta$ where $\delta=\sigma(T, x)$. Thus $\sigma(T, P_\delta y)=0$, and hence $P_\delta y=0$.

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Corollary 1. *If T is a reductive spectral operator then $T=N+Q$ where N is normal, Q is quasinilpotent and $NQ=QN$.*

Corollary 2. [3] *If T is a reductive operator which is similar to a normal operator then T is normal.*

Corollary 3. *If T is reductive and if $\sigma(T)$ (the spectrum of T) is totally disconnected then $T=N+Q$.*

Proof. If $\sigma(T)$ is totally disconnected then T is spectral if and only if T satisfies the Dunford condition (B) [1, page 2149]. Since T is reductive, the result follows from the proposition.

If T is a reductive operator and if T satisfies conditions (A) and (C), then for each closed set δ, $\sigma(T, P_\delta x)\subset\sigma(T, x)$ for all $x\in H$, where P_δ is the projection of H onto $H(\delta)$. This is the condition (I), introduced by the author in [6], where in it was shown that a decomposable operator T which satisfies condition (I), (in particular if T is reductive decomposable operatoı), then T is the sum of a normal operator and a commuting quasinilpotent operator. The next theorem is a generalization of this result.

Theorem 1. *If T satisfies conditions* (A), (C), (I), *and if for each closed set δ, $H=H(\delta)+H(\bar{\delta}')$ [$\bar{\delta}'$ denotes the closure of the complement of δ] then $T=N+Q$ where N is normal, Q is quasinilpotent, and $NQ=QN$.*

Proof. By Dunford's theorem, we only need to show that T satisfies Dunford's condition (D). In order to show this, it is enough to show that for each closed set δ, $H=H(\delta)\oplus\overline{H(\delta')}$. Since T satisfies condition (I), it satisfies the orthogonality version of condition (B) and hence $H(\delta')\subset H^\perp(\delta)$ where $H^\perp(\delta)$ is the orthogonal complement of $H(\delta)$ in H. Now for any $x\in H^\perp(\delta)$, let $x=u+v$, where $u\in H(\delta)$ and $v\in H(\bar{\delta}')$. Thus $0=P_\delta x=u+P_\delta v$ and $\sigma(T, u)=\sigma(T, P_\delta v)\subset\delta\cap\sigma(T, v)$. Hence $x\in \in H(\bar{\delta}')$. Thus for any closed set δ, $\overline{H(\delta')}\subset H^\perp(\delta)\subset H(\bar{\delta}')$. Now for any open set G which contains δ, $H^\perp(\bar{G})\subset H(\delta')$ and hence $H^\perp(\delta')\subset\bigcap H(\bar{G})=H(\delta)$, where the intersection is taken over all open sets G which contain δ.

Corollary 4. *If T is a reductive operator and if T satisfies conditions* (A) *and* (C) *and if for each closed set δ, $H=H(\delta)+H(\bar{\delta}')$ then $T=N+Q$ where N is a normal operator, Q is quasinilpotent and $NQ=QN$.*

This result appears to be a generalization of Theorem 1.1 of Jafarian [2].

Let $g: s^1=\{z: |z|=1\}\to J$ be an arc length parametrization of a rectifiable Jordan curve. Since J is rectifiable, $g'(s)$ exists almost everywhere (with respect to Lebesgue measure) on the unit circle s^1. An operator T satisfies the growth condition (G_m) if

$\|(\lambda-T)^{-1}\| \leqq M[\operatorname{dist}(\lambda, \sigma(T))]^{-m}$ for all $\lambda \in \varrho(T)$ and $|\lambda| \leqq \|T\|+1$, where $\varrho(T)$ denotes the resolvent set of T.

Theorem 2. *Let T be an operator such that $\sigma(T)$ is contained in a rectifiable Jordan curve J. If T satisfies the growth condition (G_m) and the condition* (I) *then $T=N+Q$ where N is normal operator, Q is quasinilpotent and $NQ=QN$.*

Proof. Since T satisfies the growth condition (G_m) and $\sigma(T)$ lies on J, it follows from [7, Theorem 11] that $H(\delta)$ is closed subspace of H for every closed set δ. Also from the proof of [7, Theorem 10], it follows that for any $w_1, w_2 \in J$, $w_i=g(s_i)$ are such that $g'(s_i)$ exists, $H=H[w_1, w_2]+H[w_2, w_1]$ where $[w_1, w_2]=\{g(s): s_1 \leqq s \leqq s_2\}$. By Theorem 1, we only need to show that for each closed set δ, $H=H(\delta)+H(\bar{\delta}')$, i.e., $H^{\perp}(\delta) \subset H(\bar{\delta}')$. Now suppose that there exist $x \in H^{\perp}(\delta)$ such that $\sigma(T, x) \cap \delta^0 \neq \varphi$ (δ^0 denotes the interior of δ). Then we can find $w_1, w_2 \in J$ such that $w_i=g(s_i)$ and $g'(s_i)$ exists, $\sigma(T, x) \cap [w_1, w_2] \neq \varphi$ and $[w_1, w_2] \subset \delta$. Let $x=x_1+x_2$ where $x_1 \in H[w_1, w_2]$ and $x_2 \in H[w_2, w_1]$, then $0=P_\delta x=x_1+P_\delta x_2$. Thus $\sigma(T, x_1)=\sigma(T, P_\delta x_2) \subset \delta \cap \sigma(T, x_2)$, i.e., $\sigma(T, x) \subset \delta$.

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