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Finite partitions of the real line consisting of similar sets

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In this note we shall prove and discuss a generalization of the theorem of WARREN PAGE ([3]) concerning partitions of the real line R and we shall study the Baire property of the sets in this partition.

It is not difficult to observe that if $\{A_1, \ldots, A_N\}$ is a partition of R (i.e. each A_i is nonempty, $\bigcup_{i=1}^{N} A_i = R$, and $A_i \cap A_j = \emptyset$ for $i \neq j$), then the set $G(A_1, \ldots, A_N)$ consisting of all numbers a such that $A_i + a = A_{k_i}$ for $i \in \{1, \ldots, N\}$ is an additive group. For $a \in G(A_1, \ldots, A_N)$ and $i \in \{1, \ldots, N\}$ let $f_i(a) = k_i$ if $A_i + a = A_{k_i}$. The following theorem holds:

Theorem. *If* $\{A_1, \ldots, A_N\}$, $N \geqq 2$ *is a partition of the real line such that* $G(A_1, \ldots, A_N)$ *fulfills the following conditions:*

(1) *for every* $i \in \{1, \ldots, N\}$, $f_i(G(A_1, \ldots, A_N)) = \{1, \ldots, N\}$;

(2) *for every* $i \in \{1, \ldots, N\}$, *every* $j \in f_i(G(A_1, \ldots, A_N))$, *and every* $\varepsilon > 0$ *there exists* $a \in G(A_1, \ldots, A_N)$ *such that* $|a| < \varepsilon$ *and* $A_i + a = A_j$,

then none of the sets A_i *is measurable or has the Baire property.*

Proof. Suppose that, for some $i_0 \in \{1, \ldots, N\}$, A_{i_0} is measurable. Then in virtue of (1) every A_i is measurable. Hence, similarly as in [3], from (2) we have $m(A_i \cap I) = m(A_j \cap I)$ for every i, j and for every interval I (m denotes Lebesgue measure). Then $m(A_i \cap I) = N^{-1} \cdot m(I)$ for every i and for every interval I: a contradiction with the Lebesgue density theorem. Hence each A_i is not measurable.

Suppose now that, for some $i \in \{1, \ldots, N\}$, A_{i_0} has the Baire property. Then in virtue of (1) every A_i has the Baire property. Obviously every A_i is of the second category. Let, for some i_0, $A_{i_0} = B \triangle C$, where B is open and nonempty, and C is of the first category. If (a, b) is a component of B, then $A_{i_0} \cap (a, b)$ is residual in (a, b). From (1) and (2) it follows that, for every i, $A_i \cap (a, b)$ is residual in (a, b): a contradiction. Hence none of the A_i has the Baire property. The Theorem is proved.

It is not difficult to show that for every natural $N \geqq 2$ there exists a partition $\{A_1, \ldots, A_N\}$ of R such that $G(A_1, \ldots, A_N)$ fulfills (1) and (2). We shall construct the example in a similar way as in [1]. $G_N = \{m \cdot (N+1)^{-k} \colon m, k$ — integers, $k \geqq 0\}$ is a group and $H_N = \{N \cdot m \cdot (N+1)^{-k} \colon m, k$ — integers, $k \geqq 0\}$ is a subgroup of G_N with

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index N. Let $\{C_1, \ldots, C_N\}$ be the family of all cosets of H_N in G_N. Let E be a set including exactly one number of each coset; then we set $A_i = \bigcup_{x \in E} (x+C_i)$ for $i \in \{1, \ldots, N\}$. $\{A_1, \ldots, A_N\}$ is a partition of R and $G(A_1, \ldots, A_N) = G_N$. Conditions (1) and (2) are obviously fulfilled.

If for some partition $\{A_1, \ldots, A_N\}$ the group $G(A_1, \ldots, A_N)$ fulfils only (2), then some of the sets $A_1, \ldots, A_N$, or even all of them, may be measurable and may have the Baire property. For example, if G_{N-1} and H_{N-1} are groups as above (for $N \geqq 3$), let $\{A_1, \ldots, A_{N-1}\}$ be the family of all cosets of H_{N-1} in G_{N-1} and let $A_N = R - G_{N-1}$. $\{A_1, \ldots, A_N\}$ is a partition of R, $G(A_1, \ldots, A_N) = G_{N-1}$, and all sets $A_1, \ldots, A_N$ are measurable (all but the last have measure 0) and all sets have the Baire property (all but the last are of the first category). However if we replace (1) by the following condition:

(1′) for every $i \in \{1, \ldots, N\}$, $f_i(G(A_1, \ldots, A_N))$ consists of at least two numbers,

then from (1′) and (2) it follows that in the family $\{A_1, \ldots, A_N\}$ there are at least two nonmeasurable sets and at least two sets which do not have the Baire property. The proof is similar to that of the theorem. In this case the partition may include simultaneously measurable (Baire) sets and nonmeasurable (not Baire) sets and the subfamilies of measurable sets and sets having the Baire property may be equal or not, as the following examples show: Let $\{A_1, A_2\}$ be a partition of the type constructed immediately after the proof of the theorem. Put $A_1' = H_2$, $A_2' = G_2 - H_2$, $A_3' = A_1 - G_2$, $A_4' = A_2 - G_2$. Then $\{A_1', A_2', A_3', A_4'\}$ is a partition of R fulfilling (1′) and (2), A_1' and A_2' are null sets of the first category, and A_3' and A_4' are not measurable sets which do not have the Baire property. Finally, let G_2, and H_2 be groups as above and let E be the set constructed after the proof of the theorem. If $R = A \cup B$, where A is a null set, B is of the first category, and $A \cap B = \emptyset$ (see for example [2]), set $E_1 = E \cap A$, $E_2 = E \cap B$. It is not difficult to see that E_1 and E_2 are nonempty. Let $A_1 = \bigcup_{x \in E_1} (x+H_2)$, $A_2 = \bigcup_{x \in E_1} (x+(G_2-H_2))$, $A_3 = \bigcup_{x \in E_2} (x+H_2)$, $A_4 = \bigcup_{x \in E_2} (x+(G_2-H_2))$. Then $\{A_1, A_2, A_3, A_4\}$ is a partition of R fulfilling (1′) and (2). A_1, A_2 do not have the Baire property, and A_3, A_4 are not measurable.

References

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