Acta Sci. Math., 38 (1976), 193—202

Commutants of $C_0(N)$ contractions

PEI YUAN WU

1. Introduction. Let $\mathfrak{H}$ be a complex separable Hilbert space and T a bounded linear operator on $\mathfrak{H}$. Let Lat T denote the lattice of all closed subspaces invariant under T. Let $\mathscr{A}_T$, $\{T\}''$, and $\{T\}'$ denote the smallest weakly closed subalgebra of $\mathscr{B}(\mathfrak{H})$ containing T and I, the double commutant of T, and the commutant of T, respectively. P. ROSENTHAL and D. SARASON, independently, asked the question: If $A\in\{T\}'$ and Lat $T\subset$ Lat A, is A in $\mathscr{A}_T$? An affirmative answer to this would imply affirmative answers to other unsolved problems (cf. [3]). BRICKMAN and FILLMORE [1] showed that this is true if T is an operator on a finite dimensional Hilbert space. Imitating their proof, it is not difficult to show that this also holds for algebraic operators. Recently, A. FEINTUCH [4] proved that if T is a compact operator with infinite spectrum then we also have the conclusion. In this paper we add one more class of operators to this list. We show that this holds for $C_0(N)$ contractions. We also show that such contractions are in class *(dc)* as defined in [14], that is, they satisfy $\mathscr{A}_T=\{T\}''$. Our proofs are largely dependent on the remarkable work of B. SZ.-NAGY and C. FOIAŞ on the structure of $C_0(N)$ contractions, namely, the functional models and Jordan models for such operators. A very brief description of these models will be given in § 2. The main reference for this part will be [13] and [11]. From time to time definitions and results will be taken from there without specification. § 3 contains the proofs of our main theorems.

An operator T is *reflexive* if Lat $T\subset$ Lat A implies $A\in\mathscr{A}_T$. The questions concerning reflexive operators asked by J. DEDDENS in [3] can now be answered for $C_0(N)$ contractions. These are contained in § 4, along with some characterizations for multiplicity-free contractions (cf. [10]). This provides more evidence of the analogy between $C_0(N)$ contractions and operators on finite dimensional spaces. We also give sufficient conditions for such contractions to be reflexive.

Finally, we conclude in § 5 with some remarks and open questions related to the previously given results.

Received August 5, 1975.

The author wishes to thank Dr. John B. Conway for several interesting discussions concerning some topics in this paper.

In the following **C** will denote the complex plane and **D** the open unit disk in **C**.

2. Preliminaries. Let T be a contraction on the Hilbert space $\mathfrak{H}$. T is *of class* $C_0(N)$, $N\geqq 1$, if $T^n\to 0$ and $T^{*n}\to 0$ in the strong operator topology as $n\to\infty$, and the defect indices

$$d_T \equiv \operatorname{Rank}(I-T^*T)^{1/2} \quad \text{and} \quad d_{T^*} \equiv \operatorname{Rank}(I-TT^*)^{1/2}$$

are both equal to N. Let $\Theta_T(\lambda)$ denote the characteristic function of T. Note that if T is of class $C_0(N)$, $\Theta_T(\lambda)$ is an inner function ("inner from both sides" in the terminology of [13]), that is, $\Theta_T(e^{it})$ is a unitary operator on $\mathbf{C}^N$ for almost all t. With respect to a fixed orthonormal basis of $\mathbf{C}^N$, $\Theta_T(\lambda)$ can be represented as an N by N matrix over H^∞ (the space of complex bounded analytic functions defined on **D**). Let H_N^2 denote the space of analytic functions from **D** to $\mathbf{C}^N$ which are square-integrable.

Now we assume T is a $C_0(N)$ contraction. Then T is unitarily equivalent to the compression of the shift on the space $H_N^2\ominus\Theta_T H_N^2$, that is, the operator **T** defined by

$$(\mathbf{T}^*f)(\lambda) = \frac{f(\lambda)-f(0)}{\lambda} \quad \text{for} \quad \lambda\in\mathbf{D} \quad \text{and} \quad f\in H_N^2\ominus\Theta_T H_N^2.$$

This will be called the *functional model* for T. From now on we will always consider the $C_0(N)$ contraction T as in its functional model. Moreover, to each factorization $\Theta_T(\lambda)=\Theta_2(\lambda)\Theta_1(\lambda)$ of $\Theta_T(\lambda)$ as a product of two inner functions there corresponds a subspace

$$\Theta_2 H_N^2\ominus\Theta_T H_N^2$$

invariant under T and all the invariant subspaces for T can be obtained in this way.

A contraction T is *of class* C_0 if T is completely non-unitary (c.n.u.) and there exists a function $u\neq 0$ in H^∞ such that $u(T)=0$; in this case u can be taken to be an inner function which is minimal in the sense that it will be a divisor of any inner function v for which $v(T)=0$. Such an inner function is unique up to a constant factor of modulus one; it will be called the *minimal function* of T and denoted by m_T. Note that a $C_0(N)$ contraction is of class C_0 and $\det\Theta_T$, the determinant of its characteristic function $\Theta_T(\lambda)$, is also an inner function; moreover m_T divides $\det\Theta_T$, and $\det\Theta_T$ divides m_T^N.

For a c.n.u. contraction T on $\mathfrak{H}$, a functional calculus can be defined for some functions. Indeed, let N_T denote the class of functions which are of the form $\varphi=v^{-1}u$ where $u, v\in H^\infty$ and $v(T)$ is an injective operator with dense range in $\mathfrak{H}$ (called a *quasi-affinity*); for such a function φ define

$$\varphi(T) = v(T)^{-1}u(T).$$

This definition does not depend on the particular choice of the representation $\varphi = u/v$ and, in general, $\varphi(T)$ may not be a bounded operator. If $\varphi(T)$ is a bounded operator, then $\varphi(T)$ is in the double commutant $\{T\}''$ of T. For $C_0(N)$ contractions, we have the converse:

Theorem 2.1. (see [11]) *If T is a $C_0(N)$ contraction for some $N \geqq 1$, then* $\{T\}'' \subset \{\varphi(T) \colon \varphi \in N_T\}$.

Two operators T_1 and T_2 are *quasi-similar* if there exist quasi-affinities X and Y such that

$$T_1 X = X T_2 \quad \text{and} \quad T_2 Y = Y T_1 .$$

A C_0 contraction T on H is called *multiplicity-free* if one of the following equivalent conditions holds (cf. [10] and [12]):

(i) T has a cyclic vector, i.e. a vector x_0 such that $\mathfrak{H}$ is spanned by $T^n x_0$ $(n = 0, 1, 2, \ldots)$;

(ii) T is quasi-similar to the operator $S(m_T)$ defined on $\mathfrak{H}(m_T) \equiv H^2 \ominus m_T H^2$ by

$$(S(m_T)^* f)(\lambda) = \frac{f(\lambda) - f(0)}{\lambda} \quad \text{for} \quad \lambda \in \mathbf{D} \quad \text{and} \quad f \in \mathfrak{H}(m_T).$$

Every $C_0(N)$ contraction T is quasi-similar to a uniquely determined operator of the form

$$S(m_1) \oplus S(m_2) \oplus \cdots \oplus S(m_k), \tag{2}$$

where $m_1, m_2, \ldots, m_k$ are nonconstant inner functions each of which is a divisor of its predecessor. This operator (2) is called the *Jordan model* of T. In the proof of our main theorem we will need another version of the Jordan model, which we state as

Theorem 2.2. (see [11]) *Let T be a contraction of class $C_0(N)$, $N \geqq 1$ on the space $\mathfrak{H}$. Then there exist invariant subspaces $\mathfrak{H}_1, \mathfrak{H}_2, \ldots, \mathfrak{H}_k$ for T such that $\mathfrak{H} = \bigvee_i \mathfrak{H}_i$,*

$$\Big(\bigvee_{i \in I} \mathfrak{H}_i\Big) \cap \Big(\bigvee_{j \in J} \mathfrak{H}_j\Big) = \{0\}$$

for any non-empty disjoint decomposition $\{I, J\}$ of the set $\{1, 2, \ldots, k\}$, and $T_i \equiv T | \mathfrak{H}_i$ is multiplicity-free. Moreover, if m_i is the minimal function of T_i, then m_i is a divisor of m_{i-1} for all i and m_1 coincides with the minimal function of T.

Another result needed in the sequel is the following.

Theorem 2.3. *Let T be a contraction of class $C_0(N)$ with the minimal function m_T. Let $u = u_i u_e$ be the canonical factorization of a function $u \in H^\infty$ as the product of its outer factor u_e and inner factor u_i. Then $u(T)$ is a quasi-affinity if and only if u_i and m_T have no non-trivial common inner factors.*

The proof of this theorem is essentially contained in [13] Prop. III. 4.7 (b) with minor changes; also compare [6] Theorem 2.5. We leave the details to the readers.

3. Main Theorems. A subspace $\mathfrak{K}$ is *bi-invariant* for T if $\mathfrak{K}$ is invariant under every operator in $\{T\}''$.

Theorem 3.1. *Let T be a contraction of class $C_0(N)$, $N \geqq 1$. Then every invariant subspace for T is bi-invariant.*

Proof. Let Θ_T be the characteristic function of T and consider T in its functional model as the compression of the shift on the space $\mathfrak{H} \equiv H_N^2 \ominus \Theta_T H^2$. Let $\mathfrak{K} = \Theta_2 H^2 \ominus \ominus \Theta_T H_N^2$ be an arbitrary invariant subspace for T, with the corresponding factorization

$$\Theta_T(\lambda) = \Theta_2(\lambda)\Theta_1(\lambda).$$

Let A be an operator in $\{T\}''$. Then $A = \varphi(T) = v(T)^{-1}u(T)$ for some $\varphi \in N_T$ (by Theorem 2.1).

Let $f = \Theta_2 g$ be a vector in $\mathfrak{K}$ and set $h = Af = v(T)^{-1}u(T)f$. (As all these vectors are contained in the space H_N^2 they can be considered as column N-vectors.)

We want to show that $h \in \mathfrak{K}$. Since $v(T)h = u(T)f = u(T)(\Theta_2 g)$, we have $P_H(vh) = = P_H(u\Theta_2 g)$. If follows that $vh - u\Theta_2 g \in \Theta_T H_N^2$, and hence,

$$vh = \Theta_2 w \quad \text{for some} \quad w \in H_N^2. \tag{3}$$

Carrying out the matrix multiplication and using Cramer's rule we have

$$w_j \det \Theta_2 = v \det \Phi_j, \tag{4}$$

where Φ_j is the N by N matrix obtained from Θ_2 by replacing the j-th column by the column vector h. Note that $v(T)$ is a quasi-affinity. By Theorem 2.3, v_i and m_T have no non-trivial common inner factor. As $m_T | \det \Theta_T | m_T^N$, v_i and $\det \Theta_T$, and consequently v_i and $\det \Theta_2$, have no non-trivial common inner factor, either. From (4), we conclude that v is a divisor of w_j, for $j = 1, 2, \ldots, N$. Say, $w_j = v x_j$. It is easily seen that $x_i \in H^2$ and equation (3) can be simplified to $h = \Theta_2 x$. Hence h is an element of $\Theta_2 H_N^2 \ominus \Theta_T H_N^2 = \mathfrak{K}$. This shows that $\mathfrak{K}$ is invariant under A, completing the proof.

Theorem 3.2. *Let T be a contraction of class $C_0(N)$, $N \geqq 1$. Then $\mathscr{A}_T = \{T\}''$.*

Proof. For any operator A, we denote the operator $\underbrace{A \oplus A \oplus \ldots \oplus A}_{n}$ by $A^{(n)}$. Let $A \in \{T\}''$. It is easily verified that $A^{(n)} \in \{T^{(n)}\}''$ for any $n = 1, 2, \ldots$. Note that $T^{(n)}$ is a contraction of class $C_0(nN)$. If follows from Theorem 3.1 that any invariant subspace for $T^{(n)}$ is invariant under $A^{(n)}$, that is $\operatorname{Lat} T^{(n)} \subset \operatorname{Lat} A^{(n)}$, for any n. Hence A is in $\mathscr{A}_T$ ([8] Theorem 7.1). This shows that $\{T\}'' \subset \mathscr{A}_T$. Since $\mathscr{A}_T \subset \{T\}''$ holds for any operator T, this completes the proof.

Now we are ready to prove our main theorem. The proof here is very similar to the one given by BRICKMAN and FILLMORE [1] for operators on finite dimensional spaces in that both use some kind of "Jordan model."

Theorem 3.3. *Let T be a contraction of class $C_0(N)$, $N\geqq 1$. If $A\in\{T\}'$ and* Lat $T\subset$ Lat A, *then $A\in\mathcal{A}_T$.*

Proof. Let $\mathfrak{H}_1, \mathfrak{H}_2, \dots, \mathfrak{H}_k$ be the invariant subspaces for T such that

$$\text{(5)}\qquad \text{(a)}\quad \mathfrak{H} = \bigvee_i \mathfrak{H}_i, \qquad \text{(b)}\quad \Big(\bigvee_{i\in I} \mathfrak{H}_i\Big)\cap\Big(\bigvee_{j\in J} \mathfrak{H}_j\Big) = \{0\},$$

for any decomposition $\{I, J\}$ of $\{1, 2, \dots, k\}$ and $T_i\equiv T|\mathfrak{H}_i$ is multiplicity-free with minimal function m_i satisfying $m_i|m_{i-1}$ for all i and $m_1=m_T$ (Theorem 2.2). Let $x_i\in H_i$ be a cyclic vector for T_i ($i=1, 2, \dots, k$). Consider the cyclic invariant subspace K generated by $x\equiv x_1+x_2+\dots+x_k$. We claim that the minimal function m_0 of $T_0\equiv T|\mathfrak{K}$ coincides with m_T. Indeed, since

$$m_0(T_0)x = m_0(T)x_1+\dots+m_0(T)x_k = 0,$$

by (5b) we have $m_0(T)x_1=0$. It follows that $m_0(T)\mathfrak{H}_1=0$. Hence $m_1=m_T$ is a divisor of m_0. On the other hand, since $m_T(T)\mathfrak{K}=0$, m_0 is a divisor of m_T. This shows that m_0 coincides with m_T, as asserted.

Since $\mathfrak{K}$ is invariant under T, it is also invariant under A. Let $A_0=A|\mathfrak{K}$. Since $A_0\in\{T_0\}'$ and T_0 is a multiplicity-free contraction, it is proved by SZ.-NAGY and FOIAŞ [10] that $A_0=\varphi(T_0)$ for some $\varphi\in N_{T_0}$. Say, $\varphi=\dfrac{u}{v}$, where $u, v\in H^\infty$ and $v(T_0)$ is a quasi-affinity. Hence $A_0=v(T_0)^{-1}u(T_0)$ on $\mathfrak{K}$. In particular, $v(T_0)A_0x=u(T_0)x$. Equivalently, we have

$$v(T)Ax_1+\dots+v(T)Ax_k = u(T)x_1+\dots+u(T)x_k.$$

By (5b), this implies that $v(T)Ax_i=u(T)x_i$ for all i. Hence we have $v(T)A=u(T)$ on $\mathfrak{H}_i$ for all i. It follows that $v(T)A=u(T)$ on $\mathfrak{H}$ (by (5a)). We want to show that $A\in\{T\}''$. For any $B\in\{T\}'$, we have

$$\text{(6)}\qquad v(T)AB = u(T)B = B\,u(T) = B\,v(T)A = v(T)BA.$$

Since $v(T_0)$ is a quasi-affinity on $\mathfrak{K}$, v_i and m_0 have no non-trivial common inner factor, where v_i denotes the inner factor of v (by Theorem 2.3). As shown before, m_0 coincides with m_T. Hence v_i and m_T have no non-trivial common inner factor. This implies that $v(T)$ is a quasi-affinity (by Theorem 2.3 again!). From (6), we conclude that $AB=BA$, that is $A\in\{T\}''$. On account of Theorem 3.2 the proof is done.

4. Miscellaneous results. Corollaries 4.1, 4.2 and 4.3 below answer DEDDENS' questions [3] positively for $C_0(N)$ contractions. The proofs are routine. We include them here for completeness.

Corollary 4.1. *If T is a $C_0(N)$ contraction contained in a commutative reflexive algebra $\mathscr{A}$, then T is reflexive.*

Note that a weakly closed algebra $\mathscr{A}$ is *reflexive* if $\mathscr{A}=\{A: \text{Lat}\,\mathscr{A}\subset \text{Lat}\,A\}$, where Lat $\mathscr{A}$ denotes the lattice of subspaces invariant under every operator in $\mathscr{A}$.

Proof. Let S be an operator such that Lat $T\subset$ Lat S. Since $T\in\mathscr{A}$, we have Lat $\mathscr{A}\subset$ Lat T. Hence Lat $\mathscr{A}\subset$ Lat S. The reflexivity of $\mathscr{A}$ implies that $S\in\mathscr{A}$. Hence $ST=TS$, that is, $S\in\{T\}'$. By Theorem 3.3, we conclude that $S\in\mathscr{A}_T$. This shows that T is reflexive.

Corollary 4.2. *Let T_1 and T_2 be $C_0(N)$ contractions. If T_1 and T_2 are reflexive then $T_1\oplus T_2$ is reflexive.*

Proof. Let S be an operator such that Lat $(T_1\oplus T_2)\subset$ Lat S. It is easily seen that S must be of the form $S_1\oplus S_2$, where S_1 and S_2 are operators satisfying Lat $T_1\subset$ $\subset$ Lat S_1 and Lat $T_2\subset$ Lat S_2. The reflexivity of T_1 and T_2 implies that $S_1\in\mathscr{A}_{T_1}$ and $S_2\in\mathscr{A}_{T_2}$. We have $S_1\in\{T_1\}'$ and $S_2\in\{T_2\}'$. Hence $S=S_1\oplus S_2\in\{T_1\oplus T_2\}'$. By Theorem 3.3, we conclude that $S\in\mathscr{A}_{T_1\oplus T_2}$. Hence $T_1\oplus T_2$ is reflexive, as asserted.

Corollary 4.3. *If T is a $C_0(N)$ contraction, then $T^{(n)}$ is reflexive for any $n=2, 3, \ldots$.*

Proof. We first show that $T\oplus T$ is reflexive. Let S be an operator such that Lat $(T\oplus T)\subset$ Lat S. It is easily seen that S must be of the form $S_1\oplus S_1$, where S_1 is an operator satisfying Lat $T\subset$ Lat S_1. Note that for any two operators A, B, $AB=BA$ if and only if the graph of A is an invariant subspace for $B\oplus B$. Since Lat $(T\oplus T)\subset$ Lat $(S_1\oplus S_1)$, we deduce that $S_1\in\{T\}''$. Hence $S_1\in\{T\}'$ and $S=$ $=S_1\oplus S_1\in\{T\oplus T\}'$. Using Theorem 3.3 we have $S\in\mathscr{A}_{T\oplus T}$, which shows that $T\oplus T$ is reflexive. Now we want to show that $T^{(n)}$ is reflexive for any $n\geqq 2$. Let V be an operator such that Lat $T^{(n)}\subset$ Lat V. As before, we have $V=V_1^{(n)}$ for some operator V_1 satisfying Lat $T\subset$ Lat V_1. From Lat $T^{(n)}\subset$ Lat $V_1^{(n)}$ we deduce that Lat $T^{(2)}\subset$ $\subset$ Lat $V_1^{(2)}$. By what we just proved, $V^{(2)}\in\mathscr{A}_{T^{(2)}}\subset\{T^{(2)}\}'$. Hence $V_1\in\{T\}'$ and $V_1^{(n)}\in\{T^{(n)}\}'$. It follows from Theorem 3.3 that $V=V_1^{(n)}\in\mathscr{A}_{T^{(n)}}$. Hence $T^{(n)}$ is reflexive, completing the proof.

It was proved by Sz.-Nagy and Foiaş that a $C_0(N)$ contraction T is multiplicity-free if and only if $\{T\}'$ is abelian, or equivalently, $\{T\}''=\{T\}'$. (This and other characterizations can be found in [10] and [12].) The next corollary gives some other equivalent conditions.

Corollary 4.4. *Let T be a contraction of class $C_0(N)$, $N\geqq 1$. Then the following are equivalent to each other:*

(i) *T is multiplicity-free;*

(ii) *$\mathscr{A}_T=\{T\}'$;*

(iii) *$\{T\}'$ is a singly generated algebra;*

(iv) *$\{T\}'$ is a maximal abelian algebra, that is, $\{T\}'$ is abelian and if $\mathscr{A}$ is a weakly closed abelian algebra containing $\{T\}'$, then $\mathscr{A}=\{T\}'$;*

(v) *Every invariant subspace for T is hyperinvariant, that is, invariant under every operator in $\{T\}'$.*

If this is the case, then $\mathscr{A}_T=\{T\}''=\{T\}'\subset\{\varphi(T): \varphi\in N_T\}$.

Proof. That (i) implies (ii) follows from Theorem 3.2 and the remark given above; (v) implies (ii) follows from Theorem 3.3. Other implications are clear.

It seems to be unknown whether reflexive operators are preserved under quasi-similarities. (Note that they are preserved under similarities.) The next corollary makes a modest step in this direction.

Corollary 4.5. *Let T_1 and T_2 be $C_0(N)$ contractions which are multiplicity-free. Assume T_1 is quasi-similar to T_2. Then T_1 is reflexive if and only if T_2 is.*

Proof. By symmetry, we have only to show half of the assertion. Assume T_1 is reflexive. Let X and Y be quasi-affinities such that $T_1X=XT_2$ and $T_2Y=YT_1$. Let S be an operator with Lat $T_2\subset$ Lat S, and $\mathfrak{K}_1$ be an invariant subspace for T_1. Assume m is the minimal function of $T_1|\mathfrak{K}_1$. Let $\mathfrak{K}_2$ be the unique invariant subspace for T_2 for which $T_2|\mathfrak{K}_2$ has minimal function m (cf. [10]). Note that $\mathfrak{K}_1=\{x: m(T_1)x=0\}$ and $\mathfrak{K}_2=\{y: m(T_2)y=0\}$ ([10]). For any $x\in\mathfrak{K}_1$, we have $m(T_2)Yx=Ym(T_1)x=0$. This implies that $Yx\in\mathfrak{K}_2$. Since $\mathfrak{K}_2$ is invariant for S, we have $SYx\in\mathfrak{K}_2$. Hence $m(T_1)XSYx=Xm(T_2)SYx=0$. This shows that $XSYx\in\mathfrak{K}_1$, and hence $\mathfrak{K}_1$ is invariant under XSY. Since $\mathfrak{K}_1$ is arbitrary, we conclude that $XSY\in\mathscr{A}_{T_1}$ (by the reflexivity of T_1). In particular, XSY commutes with T_1. Since X, Y are quasi-affinities, it is easily seen that S must commute with T_2. Using Theorem 3.3, we have $S\in\mathscr{A}_{T_2}$. This shows that T_2 is reflexive, completing the proof.

As a special case, we have

Corollary 4.6. *Let φ_1, φ_2 be (scalar valued) inner functions with $(\varphi_1, \varphi_2)=1$, and $\varphi=\varphi_1\cdot\varphi_2$. Let $S(\varphi_1)$, $S(\varphi_2)$ and $S(\varphi)$ denote the corresponding compressions of the shift acting on $\mathfrak{H}(\varphi_1)$, $\mathfrak{H}(\varphi_2)$ and $\mathfrak{H}(\varphi)$, respectively. Then the following are equivalent to each other:*

(i) *$S(\varphi_1)$ and $S(\varphi_2)$ are reflexive;*

(ii) *$S(\varphi_1)\oplus S(\varphi_2)$ is reflexive;*

(iii) *$S(\varphi)$ is reflexive.*

Proof. The equivalence of (i) and (ii) is proved in [2]. The equivalence of (ii) and (iii) follows from Corollary 4.5 and the fact that $S(\varphi_1)\oplus S(\varphi_2)$ and $S(\varphi)$ are quasi-similar to each other for relatively prime inner functions φ_1, φ_2 (cf. [9], pp. 50—51).

J. ERDŐS has asked whether operators with the property that their invariant subspaces are all spanned by eigenvectors are necessarily reflexive. The next corollary answers the question positively for $C_0(N)$ contractions. Note that for such contractions, that all invariant subspaces are spanned by eigenvectors is equivalent to the fact that the minimal function is a Blaschke product with simple zeros (cf. [13], Prop. III. 7.2).

Corollary 4.7. *If T is a $C_0(N)$ contraction on $\mathfrak{H}$ whose minimal function m_T is a Blaschke product with simple zeros, then T is reflexive.*

Proof. Let S be an operator such that $\operatorname{Lat} T \subset \operatorname{Lat} S$. Let $\{\lambda_i\}$ be the zeros of m_T. Then $\{\lambda_i\}$ are eigenvalues for T. If $\mathfrak{H}_i$ denotes the subspace of eigenvectors associated with λ_i, then $\mathfrak{H}_i$ $(i=1, 2, \ldots)$ span $\mathfrak{H}$ (cf. [13], Prop. III. 7.2). Each $\mathfrak{H}_i$, being invariant for T, is invariant under S. Hence for $x_i \in \mathfrak{H}_i$ we have

$$TSx_i = \lambda_i Sx_i = S\lambda_i x_i = STx_i.$$

This shows that T and S commute on $\mathfrak{H}_i$, $i=1, 2, \ldots$. It follows that $TS=ST$ on $\mathfrak{H}$. By Theorem 3.3, we have $S \in \mathscr{A}_T$. Hence T is reflexive, as asserted.

Note that the condition we give here is, in general, not necessary. As an example, consider the operator $S(\varphi) \oplus S(\varphi)$, which is reflexive for any inner function φ (by Corollary 4.3). However, for compressions of the shift we have

Corollary 4.8. *Let φ be a Blaschke product and $S(\varphi)$ the corresponding compression of the shift. Then $S(\varphi)$ is reflexive if and only if φ has only simple zeros.*

In the proof we will need the following simple fact, due to DEDDENS [3], concerning unicellular operators, the proof of which is included here for completeness. Recall that an operator T is *unicellular* if $\operatorname{Lat} T$ is totally ordered.

Lemma 4.9. *No unicellular operator on a space with dimension $\geqq 2$ is reflexive.*

Proof. Let T be a unicellular operator acting on $\mathfrak{H}$ which is reflexive. Let $\mathfrak{H}_1$, $\mathfrak{H}_2$ be invariant subspaces for T and P_1 the (orthogonal) projection onto $\mathfrak{H}_1$. We have $\mathfrak{H}_1 \subset \mathfrak{H}_2$ or $\mathfrak{H}_2 \subset \mathfrak{H}_1$. In either case P_1 will leave $\mathfrak{H}_2$ invariant. Since $\mathfrak{H}_2$ is arbitrary, by the reflexivity of T we have $P_1 \in \mathscr{A}_T$. Thus P_1 commutes with T. Hence both $\mathfrak{H}_1$ and $\mathfrak{H}_1^{\perp}$ are invariant under T. Then $\mathfrak{H}_1 \subset \mathfrak{H}_1^{\perp}$ or $\mathfrak{H}_1^{\perp} \subset \mathfrak{H}$ and we have $\mathfrak{H}_1 = \{0\}$ or $\mathfrak{H}_1 = \mathfrak{H}$. This shows that the only invariant subspaces for T are $\{0\}$ and $\mathfrak{H}$. Thus every operator on $\mathfrak{H}$ is in $\mathscr{A}_T$, hence commutes with T. A standard argument shows that T is a scalar multiple of the identity. Obviously, this cannot happen unless $\dim \mathfrak{H} = 0$ or 1, which proves our assertion.

Proof of Corollary 4.8. We have only to show that if $S(\varphi)$ is reflexive then φ has only simple zeros. Assume that λ_0 is a zero of φ with multiplicity $n_0 \geqq 2$. We have $\varphi(\lambda)=\varphi_0(\lambda)\varphi_1(\lambda)$, where

$$\varphi_0(\lambda) = \left(\frac{\lambda-\lambda_0}{1-\bar{\lambda}_0\lambda}\right)^{n_0} \quad \text{for} \quad \lambda \in \mathbf{D},$$

and $\varphi_1(\lambda)$ is a Blaschke product with $\varphi_1(\lambda_0)\neq 0$. Since $(\varphi_0, \varphi_1)=1$, the reflexivity of $S(\varphi)$ implies the reflexivity of $S(\varphi_0)$ (by Corollary 4.6). But it is easily seen that $S(\varphi_0)$ is a unicellular operator on a space with dimension $n_0 \geqq 2$. By Lemma 4.9 we have a contradiction, which proves our assertion.

Consider an inner function $\varphi(\lambda)=\psi(\lambda)\eta(\lambda)$ factored as the product of a Blaschke product $\psi(\lambda)$ and a singular inner function $\eta(\lambda)$. (For the structure of scalar valued inner functions consult [7].) Since $(\psi, \eta)=1$, we conclude, by Corollary 4.6, that $S(\varphi)$ is reflexive if and only if $S(\psi)$ and $S(\eta)$ both are reflexive. The preceding corollary gives a complete characterization for $S(\psi)$ being reflexive. As for the case of $S(\eta)$, we are not so fortunate. We have only the following partial result.

Recall that a *singular inner function* $\eta(\lambda)$ is a function of the form

$$\eta(\lambda) \equiv \eta(\mu; \lambda) = \exp\left(-\int \frac{e^{it}+\lambda}{e^{it}-\lambda}\, d\mu(t)\right),$$

where μ is a finite positive Borel measure on the unit circle C which is singular with respect to Lebesgue measure. The measure μ has an *atom* E, if E is a Borel subset with $\mu(E)>0$ and for any Borel subset F of E we have $\mu(F)=0$ or $\mu(E\setminus F)=0$.

Corollary 4.10. *If η is a singular inner function whose associated measure μ has an atom, then $S(\eta)$ is not reflexive.*

Proof. Let E be an atom of μ. Consider the functions $\eta_E(\lambda)=\eta(\mu_E; \lambda)$ and $\eta_{C\setminus E}(\lambda)=\eta(\mu_{C\setminus E}; \lambda)$, where μ_E and $\mu_{C\setminus E}$ are the restrictions of the measure μ to the sets E and $C\setminus E$, respectively. Note that $\eta(\lambda)=\eta_E(\lambda)\eta_{C\setminus E}(\lambda)$ and $(\eta_E, \eta_{C\setminus E})=1$. If $S(\eta)$ is reflexive, so is $S(\eta_E)$ (by Corollary 4.6). As any inner factor of $\eta_E(\mu; \lambda)$ must be of the form $\eta_E(a\mu; \lambda)$ for some $a\in[0, 1]$, the lattice of invariant subspaces of $S(\eta_E)$ is totally ordered, that is, $S(\eta_E)$ is unicellular. By Lemma 4.9 this can happen only when the space on which $S(\eta_E)$ is acting has dimension $\leqq 1$. However, this is impossible for a singular inner function η_E. This shows that $S(\eta)$ cannot be reflexive.

Note that the preceding result does not hold for $C_0(N)$ contractions with singular, atomic minimal functions. (Consider the direct sum of a compression of the shift with itself.) On the other hand, whether $C_0(N)$ contractions with singular, totally non-atomic minimal functions are indeed reflexive is still unknown. C. Foiaş [5] has shown that $S(\varphi)$ is reflexive for certain singular φ with totally non-atomic measures.

We remark that Corollaries 4.8 and 4.10 have been obtained earlier by J. CONWAY and, independently, by B. MOORE, III and E. NORDGREN (unpublished).

5. Concluding remarks. As the Jordan models for $C_0(N)$ contractions have been generalized to C_0 contractions with finite multiplicity (cf. [12]), it seems likely that our main theorems in § 3 hold in this more general context. However the proofs we gave do not seem to be readily extended to cover this case.

We also remark that if the answer to Rosenthal and Sarason's question (cf. § 1) is affirmative, most of the results we gave in § 4 will hold for arbitrary operators.

Finally, we raise the following question to conclude this paper: If T_1 and T_2 are $C_0(N)$ contractions which are quasi-similar to each other, is it true that T_1 is reflexive if and only if T_2 is? (The answer is "yes" for $C_0(N)$ contractions which are multiplicity-free.)

Bibliography

[1] L. BRICKMAN and P. A. FILLMORE, The invariant subspace lattice of a linear transformation, *Canadian J. Math.*, **19** (1967), 810—822.
[2] J. B. CONWAY and P. Y. WU, The splitting of $\mathscr{A}(T_1 \oplus T_2)$ and related questions, *to appear.*
[3] J. A. DEDDENS, Reflexive operators, *Indiana Univ. Math. J.*, **20** (1971), 887—889.
[4] A. FEINTUCH, On commutants of compact operators, *Duke Math. J.*, **41** (1974), 387—391.
[5] C. FOIAŞ, On the scalar parts of a decomposable operator, *Rev. Roum. Math. Pures et Appl.*, **17** (1972), 1181—1198.
[6] P. A. FUHRMANN, On the corona theorem and its application to spectral problems in Hilbert space, *Trans. Amer. Math. Soc.*, **132** (1968), 55—66.
[7] K. HOFFMAN, *Banach spaces of analytic functions*, Prentice-Hall (Englewood Cliffs, N. J., 1962).
[8] H. RADJAVI and P. ROSENTHAL, *Invariant subspaces*, Springer-Verlag (Berlin/Heidelberg/New York, 1973).
[9] B. SZ.-NAGY, *Unitary dilations of Hilbert space operators and related topics*, Regional Conference series in Mathematics, No. 19, Amer. Math. Soc. (Providence, R. I., 1974).
[10] B. SZ.-NAGY and C. FOIAŞ, Opérateurs sans multiplicité, *Acta Sci. Math.*, **30** (1969), 1—18.
[11] B. SZ.-NAGY and C. FOIAŞ, Modèles de Jordan pour une classe d'opérateurs de l'espace de Hilbert, *Acta Sci. Math.*, **31** (1970), 91—115.
[12] B. SZ.-NAGY and C. FOIAŞ, Compléments à l'étude des opérateurs de classe C_0, *Acta Sci. Math.*, **31** (1970), 287—296.
[13] B. SZ.-NAGY and C. FOIAŞ, *Harmonic analysis of operators on Hilbert space*, North Holland/Akadémiai Kiadó (Amsterdam/Budapest, 1970).
[14] T. R. TURNER, Double commutants of algebraic operators, *Proc. Amer. Math. Soc.*, **33** (1972), 415—419.

NATIONAL CHIAO TUNG UNIVERSITY
HSINCHU, TAIWAN, CHINA