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Strongly reductive operators are normal

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An operator on a Hilbert space $\mathfrak{H}$ is called reductive if every subspace $\mathfrak{L}$ [1]) invariant for T is also invariant for T^* (i.e. $\mathfrak{L}$ is reducing T). By a theorem of DYER, PEDERSEN and PORCELLI [4] every reductive operator is normal if and only if every operator has a (non-trivial) invariant subspace. Therefore the study of reductive operators might shed some light into the intricate structure of general operators. In particular it looked instructive to study a natural subclass of reductive operators [6], [2]. Let us recall that an operator T on $\mathfrak{H}$ is called *strongly reductive* if

$$\varepsilon_T(\delta) = \sup\{\|(I-P)T^*P\| : \|(I-P)TP\| < \delta\}$$

tends to 0 for $\delta \searrow 0$; P runs through the family $\mathscr{P}_{\mathfrak{H}}$ of orthogonal projections in $\mathfrak{H}$. Concerning this concept, the following was proved by HARRISON [6] (Cor. 2.4. and Thm. 3.8).

Proposition. *If T is strongly reductive then its spectrum $\sigma(T)$ neither divides the (complex) plane nor has interior (in the plane). These conditions on $\sigma(T)$ imply, in case T is normal, that T is strongly reductive.*

The aim of this short Note is to supplement these results with the following.

Theorem. *Every strongly reductive operator is normal.*

We will divide the proof of this theorem in several steps:

1. Lemma. *Let T be a strongly reductive operator on $\mathfrak{H}$ and let X be an operator on some space $\mathfrak{K}$ such that $\|X-U_jTU_j^{-1}\|\to 0$ $(j\to\infty)$ where U_j $(j=1,2,\ldots)$ are unitary operators from $\mathfrak{H}$ onto $\mathfrak{K}$. Then X is also strongly reductive.*

Proof. For $\delta>0$ let $P\in\mathscr{P}_{\mathfrak{K}}$ be such that $\|(I-P)XP\|<\delta$. Denote $T_j=U_jTU_j^{-1}$ and take j large enough such that $\|X-T_j\|<\delta-\|(I-P)XP\|$. Then for $P_j=U_j^{-1}PU_j$

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[1]) All the spaces involved are complex Hilbert spaces; the subspaces will be always considered linear and closed. Also all operators will be linear, continuous, and mapping Hilbert spaces into Hilbert spaces.

we have $P_j \in \mathscr{P}_{\mathfrak{H}}$ and $\|(I-P_j)TP_j\| < \delta$ so that

$$\|(I-P)X^*P\| \leqq \|X^*-T_j^*\| + \|(I-P)T_j^*P\| =$$
$$= \|X-T_j\| + \|(I-P_j)T^*P_j\| \leqq \|X-T_j\| + \varepsilon_T(\delta),$$

whence (letting $j \to \infty$), $\|(I-P)X^*P\| \leqq \varepsilon_T(\delta)$.

2. Lemma. *Let T be a strongly reductive operator on a separable space $\mathfrak{H}$. Then T^*T-TT^* is compact.*

Proof. Let $\mathscr{B}$ be the C^*-algebra with unity, generated in the Calkin algebra $C(\mathfrak{H})$ [2]) by the image $\tilde{T}$ of T. Let moreover ϱ be a faithful C^*-representation of $\mathscr{B}$ on a separable Hilbert space $\mathfrak{H}_\varrho$. [3]) By virtue of [8], Thm. 1.3, we can take the operator X in Lemma 1 of the form $X = T \oplus \varrho(\tilde{T}) \oplus \varrho(\tilde{T})$; therefore this operator is strongly reductive, henceforth reductive. But if P denotes the orthogonal projection of $\mathfrak{H} \oplus \mathfrak{H}_\varrho \oplus \mathfrak{H}_\varrho$ onto $\{0 \oplus h \oplus \varrho(\tilde{T})h : h \in \mathfrak{H}_\varrho\}$ then $(I-P)XP = 0$, thus also $\|(I-P)X^*P\| = 0$. Whence we easily infer that $\varrho(\tilde{T})^*\varrho(\tilde{T})h = \varrho(\tilde{T})\varrho(\tilde{T})^*h$ for all $h \in \mathfrak{H}_\varrho$, i.e. $\varrho(\tilde{T}^*\tilde{T} - \tilde{T}\tilde{T}^*) = 0$, $\widetilde{T^*T - TT^*} = \tilde{T}^*\tilde{T} - \tilde{T}\tilde{T}^* = 0$.

3. Lemma. *Let T be a strongly reductive operator on $\mathfrak{H}$. Then, if* $\dim \mathfrak{H} > 1$, *there exists a (non-trivial) subspace of $\mathfrak{H}$, invariant for T (thus also reducing T).*

Proof. Since, if $\dim \mathfrak{H} < \infty$ then T is obviously normal and if $\dim \mathfrak{H} > \aleph_0$ then T is obviously reduced by separable subspaces of $\mathfrak{H}$, it remains to consider only the case $\dim \mathfrak{H} = \aleph_0$. In this case, the properties of $\sigma(T)$ (yielded by Harrison's Proposition) together with the spectral characterization of quasitriangular operators [3], Thm. 5.4, imply that T is quasi-triangular. Therefore if $\|p(T)\| \neq \|\widetilde{p(T)}\|$ for some polynomial $p(\lambda)$, the existence of (non-trivial) subspaces reducing T is already established in [2]. Thus we can assume that

$$(1) \qquad \|p(T)\| = \|\widetilde{p(T)}\| = \|p(\tilde{T})\|$$

for all polynomials $p(\lambda)$. But in virtue of Lemma 2, T is normal in $C(\mathfrak{H})$, thus

$$(2) \qquad \|p(T)\| = \|p\|_{C(\sigma(\tilde{T}))} (:= \max\{|p(\lambda)| : \lambda \in \sigma(\tilde{T})\}),$$

where $\sigma(\tilde{T}) (\subset \sigma(T))$ neither separates the plane nor has interior. By (1), (2) and by virtue of the classical theorem of Lavrentiev [5], Ch. II, 8.7, the map $p|_{\sigma(\tilde{T})} \mapsto p(T)$ extends to an isometric algebraic map of $C(\sigma(\tilde{T}))$ in $L(\mathfrak{H})$. Consequently, if $\sigma(\tilde{T})$

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[2]) This is the quotient C^*-algebra $C(\mathfrak{H}) = L(\mathfrak{H})/K(\mathfrak{H})$, where $L(\mathfrak{H})$ denotes the algebra of all operators on $\mathfrak{H}$ while $K(\mathfrak{H})$ denotes the ideal of all compact operators on $\mathfrak{H}$. We shall denote the element $X + K(\mathfrak{H})$ $(X \in L(\mathfrak{H}))$ in $C(\mathfrak{H})$ by $\tilde{X}$.

[3]) The existence of such a representation follows easily from the separability of $\mathscr{B}$ and the classical Gelfand — Naĭmark theorem [7], Ch. V., § 24, Sec. 2.

reduces to a single point λ, then $T=\lambda$, if not then taking two continuous functions f and g on $\sigma(\tilde{T})$ not vanishing identically and such that $fg=0$ we have $f(T)\neq 0$, $g(T)\neq 0$, $f(T)g(T)=0$, and T leaves invariant the (non-trivial) null-spaces of $f(T)$ and $g(T)$.

4. We are now in state to achieve the proof of the theorem. As in the proof of Lemma 3 we can assume that $\mathfrak{H}$ is separable. Also we can discard from $\mathfrak{H}$ the largest reducing subspace $\mathfrak{L}$ of $\mathfrak{H}$ on which $T|\mathfrak{L}$ is normal (see [1]). Therefore, in case T is not normal we can assume that for any subspace $\mathfrak{L}\subset\mathfrak{H}$, reducing T, the operator $T|\mathfrak{L}$ is not normal; it follows that for such subspaces $\mathfrak{L}$ we have $\dim\mathfrak{L}=\aleph_0$. Using these facts together with Lemma 3 we can prove that for any maximal totally ordered family $\mathcal{F}$ of invariant subspaces $\mathfrak{K}$ for T and for every $\mathfrak{K}_0\in\mathcal{F}$ the continuity properties

$$\bigvee\{\mathfrak{K}:\mathfrak{K}\subsetneqq\mathfrak{K}_0,\ \mathfrak{K}\in\mathcal{F}\}=\mathfrak{K}_0=\bigcap\{\mathfrak{K}:\mathfrak{K}\supsetneqq\mathfrak{K}_0,\ \mathfrak{K}\in\mathcal{F}\}$$

hold. Moreover, $\{0\}$ and $\mathfrak{H}$ belong to $\mathcal{F}$. As T is (strongly) reductive the subspaces $\mathfrak{K}$ reduce T, and therefore, $C=T^*T-TT^*$ too. Since T is not normal, $C\neq 0$. On the other hand, by Lemma 2 the operator C is compact so that it has a finite dimensional non-zero eigen-subspace $\mathfrak{L}$. Then the corresponding orthogonal projection $P_{\mathfrak{L}}$ is reduced by each $\mathfrak{K}\in\mathcal{F}$. Consequently, $\mathcal{F}'=\{\mathfrak{K}\cap\mathfrak{L}:\ \mathfrak{K}\in\mathcal{F}\}$ has the same continuity properties as $\mathcal{F}$. This contradicts the finite dimensionality of $\mathfrak{L}$.

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