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Jordan model for some operators

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The aim of this Note is to find the Jordan model of a C_0 operator whose characteristic function coincides with $e_A(z)=\exp\left(A\frac{z+1}{z-1}\right)$, where A is a bounded positive operator acting on a separable Hilbert space $\mathfrak{K}$. This problem was proposed by C. Foiaş for $\mathfrak{K}=L^2(0,1)$ and the operator A defined by $(Af)(x)=xf(x)$, $f\in L^2(0,1)$.

1. Preliminaries

We will frequently use the following assertion. If T, T' are two quasisimilar completely non-unitary contractions, $m\in H^\infty$, $\mathfrak{N}=(\operatorname{ran} m(T))^-$ and $\mathfrak{N}'=(\operatorname{ran} m(T'))^-$, then $T|\mathfrak{N}$ and $T'|\mathfrak{N}'$ are also quasisimilar (cf. [2]).

Let us recall that if the operator T is acting on $\mathfrak{H}$, its multiplicity μ_T is defined as the minimum cardinality of a subset $\mathfrak{M}\subset\mathfrak{H}$ such that $\bigvee_{n=0}^{\infty} T^n\mathfrak{M}=\mathfrak{H}$. If T and T' are quasisimilar, then $\mu_T=\mu_{T'}$ (cf. [3]).

Proposition A. (cf. [4], [5], [1]) *Let T be a C_0 operator acting on a separable Hilbert space. Then there exists a sequence $\{m_j\}_{j=1}^n$ of inner functions such that:*

(1) *m_{j+1} divides m_j for each j;*

(2) *T is quasisimilar to $\bigoplus_{j=1}^{n} S(m_j)$;*

(3) $m_1=m_T$;

(4) $n=\mu_T$ ($\leqq\infty$).

The sequence $\{m_j\}_{j=1}^n$ is uniquely determined by conditions (1) *and* (2).

The operator $\bigoplus_{j=1}^{n} S(m_j)$ is called the Jordan model of T. An operator of the form $\bigoplus_{j=1}^{n} S(m_j)$, for which (1) holds, is called a Jordan operator.

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Let us recall that with each inner function $\{\mathfrak{K}, \mathfrak{K}, \Theta(z)\}$ in the unit disc we can associate the operator $S(\Theta)$ acting on the space

$$\mathfrak{H}(\Theta) = H^2(\mathfrak{K}) \ominus \Theta H^2(\mathfrak{K}), \tag{1.1}$$

defined by

$$S(\Theta)u = P_{\mathfrak{H}(\Theta)}(zu(z)), \quad u \in \mathfrak{H}(\Theta). \tag{1.2}$$

If the function $\{\mathfrak{K}, \mathfrak{K}, \Theta(z)\}$ is pure, then it coincides with the characteristic function of the contraction $S(\Theta)$ (cf. [2]).

If is obvious that if I is an at most countable set and for each $i \in I$, $\{\mathfrak{K}_i, \mathfrak{K}_i, \Theta_i(z)\}$ is an inner function in the unit disc, then the function $\{\mathfrak{K}, \mathfrak{K}, \Theta(z)\}$, where $\mathfrak{K} = \bigoplus_{i \in I} \mathfrak{K}_i$ and $\Theta(z) = \bigoplus_{i \in I} \Theta_i(z)$, is also inner and we have

$$S(\Theta) = \bigoplus_{i \in I} S(\Theta_i). \tag{1.3}$$

2. The Jordan model of $S(e_A)$

Let A be a positive operator on the separable Hilbert space $\mathfrak{K}$, with spectral measure E. We can then define an inner function $\{\mathfrak{K}, \mathfrak{K}, e_A(z)\}$ by the formula:

$$e_A(z) = \exp\left(A\frac{z+1}{z-1}\right) = \int_0^a e_t(z)\,dE_t, \quad a = \|A\|, \tag{2.1}$$

where we use the notation:

$$e_t(z) = \exp\left(t\frac{z+1}{z-1}\right). \tag{2.2}$$

As $e_A(0) = \exp(-A)$, it is easy to see that the function e_A is pure if and only if $\ker A = \{0\}$.

Lemma 1. *The characteristic function of*

$$S(e_A)\,|\,(\operatorname{ran} e_t(S(e_A)))^-, \quad t \geqq 0,$$

is $\{\mathfrak{K}_t, \mathfrak{K}_t, e_{A_t}(z)\}$, *where* $\mathfrak{K}_t = E((t, \|A\|])\mathfrak{K}$ *and* $A_t = (A - tI)|\mathfrak{K}_t$. *Thus* $S(e_A)$ *is a* C_0 *operator and its minimal function is* $e_{\|A\|}$.

Proof. We first show that

$$(\operatorname{ran} e_t(S(e_A)))^- = e_{A_t'}H^2(\mathfrak{K}) \ominus e_A H^2(\mathfrak{K}) \tag{2.3}$$

where

$$A_t' = AE((0, t]) + tE((t, \|A\|]). \tag{2.4}$$

Indeed we have

$$(2.5)\qquad \begin{aligned}(\operatorname{ran} e_t(S(e_A)))^- &= (P_{\mathfrak{H}(e_A)} e_t \mathfrak{H}(e_A))^- = (P_{\mathfrak{H}(e_A)} e_t H^2(\mathfrak{K}))^- = \\ &= (e_t H^2(\mathfrak{K}) + e_A H^2(\mathfrak{K}))^- \ominus e_A H^2(\mathfrak{K}).\end{aligned}$$

The operator of multiplication by e_t on $H^2(\mathfrak{K})$ may be represented as a product $e_{A_t'} e_{A_t''}$, where $A_t'' = (tI - A)E((0, t])$, thus $e_t H^2(\mathfrak{K}) \subset e_{A_t'} H^2(\mathfrak{K})$ and from (2.5) we infer

$$(2.6)\qquad (\operatorname{ran} e_t(S(e_A)))^- \subset e_{A_t'} H^2(\mathfrak{K}) \ominus e_A H^2(\mathfrak{K}).$$

Now, for $u \in H^2(\mathfrak{K})$ we have

$$e_{A_t'} u = e_A E((0, t]) u + e_t E((t, \|A\|]) u,$$

thus $e_{A_t'} H^2(\mathfrak{K}) \subset e_A H^2(\mathfrak{K}) + e_t H^2(\mathfrak{K})$ and from (2.5) we infer

$$e_{A_t'} H^2(\mathfrak{K}) \ominus e_A H^2(\mathfrak{K}) \subset (\operatorname{ran} e_t(S(e_A)))^-.$$

This inclusion and (2.6) prove the equality (2.3).

Now let us remark that the operator $R\colon \mathfrak{H}(e_{A_t}) \to \mathfrak{H}(e_A)$ defined by $Ru = e_t u$ is isometric,

$$R\mathfrak{H}(e_{A_t}) = e_t H^2(\mathfrak{K}_t) \ominus e_A H^2(\mathfrak{K}_t) = e_{A_t'} H^2(\mathfrak{K}) \ominus e_A H^2(\mathfrak{K}) = (\operatorname{ran} e_t(S(e_A)))^-$$

and $RS(e_{A_t}) = S(e_A)R$. Thus $S(e_A)|(\operatorname{ran} e_t(S(e_A)))^-$ is unitarily equivalent so $S(e_{A_t})$ and the lemma follows if we remark that $\ker A_t = \{0\}$, that is e_{A_t} is pure.

Lemma 2. *We have* $\mu_{S(e_A)} = \operatorname{Rank} A$.

Proof. We may suppose without loss of generality that $\ker A = \{0\}$. If $\operatorname{Rank} A = n < \infty$, A is represented, for an adequate choice of the basis in $\mathfrak{K}$, by the matrix

$$\begin{pmatrix} t_1 & 0 & \dots & 0 \\ 0 & t_2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & t_n \end{pmatrix}, \quad t_1 \geqq t_2 \geqq \dots \geqq t_n > 0.$$

It follows that $S(e_A)$ is unitarily equivalent to the Jordan operator $\bigoplus_{j=1}^{n} S(e_{t_j})$; thus $S(e_A)$ is of multiplicity n.

Conversely, let us suppose that $S(e_A)$ is of multiplicity $n < \infty$. We show first that the spectrum $\sigma(A)$ consists of at most n points. If $\sigma(A)$ contains more than n points, we can find $0 = t_0 < t_1 < \dots < t_{n+1} = \|A\|$ such that $E((t_i, t_{i+1}]) \neq 0$, $i = 0, 1, \dots, n$. Because $A = \bigoplus_{i=0}^{n} A|E((t_i, t_{i+1}])\mathfrak{K} = \bigoplus_{i=0}^{n} A_i$, we have $S(e_A) = \bigoplus_{i=0}^{n} S(e_{A_i})$. From Lemma 1 and Proposition A it follows that $S(e_{A_i})$ is quasisimilar to a Jordan operator

$$S(e_{s_i}) \oplus \dots, \quad \text{where} \quad s_i = \|A_i\| \in (t_i, t_{i+1}].$$

Thus $S(e_A)$ is quasisimilar to

$$T = S(e_{s_n}) \oplus S(e_{s_{n-1}}) \oplus \dots \oplus S(e_{s_0}) \oplus \dots,$$

$s_n > s_{n-1} > \ldots > s_0 > 0$ (T may not be a Jordan operator). It is clear that $\mu_T \geqq n+1$ and this contradicts the equality $\mu_T = \mu_{S(e_A)} = n$. Thus $\sigma(A)$ consits of at most n points, say

$$\sigma(A) = \{\tau_1, \tau_2, \ldots, \tau_k\}, \quad \tau_1 > \tau_2 > \ldots > \tau_k > 0 \quad (k \leqq n).$$

Each τ_i is an eigenvalue of A say of multiplicity $n_i (\leqq \infty)$. Because $A = \bigoplus_{i=1}^{k} A|E(\{\tau_i\})\mathfrak{R}$, it follows that $S(e_A)$ is unitarily equivalent to

$$\bigoplus_{i=1}^{k} \left(\bigoplus_{j=1}^{n_i} S(e_{\tau_i}) \right). \tag{2.7}$$

Now, the operator (2.7) is of finite multiplicity if and only if $n_i < \infty$, $i = 1, \ldots, k$, and then its multiplicity equals $n_1 + n_2 + \ldots + n_k = \text{Rank } A$. The lemma follows.

Lemma 3. *Let $S = \bigoplus_{j=1}^{\infty} S(m_j)$ be a Jordan operator of infinite multiplicity and let T be a C_0 operator acting on a separable Hilbert space with the property that m_T divides m_j for each j. Then the Jordan model of $T \oplus S$ is S.*

Proof. Let $S' = \bigoplus_{j=1}^{\infty} S(m'_j)$ be the Jordan model of $T \oplus S$. For each j, $(T \oplus S)|(\text{ran } m'_j(T \oplus S))^-$ is quasisimilar to $S'|(\text{ran } m'_j(S'))^-$, thus it has finite multiplicity. It follows that, for sufficiently large i, $m'_j(S(m_i)) = 0$, thus m_i divides m'_j. From the hypothesis it follows that m_T divides m'_j for each j. Now, $(T \oplus S)|(\text{ran } m_T(T \oplus S))^-$ and $S'|(\text{ran } m_T(S'))^-$ are quasisimilar. Because $(T \oplus S)|(\text{ran } m_T(T \oplus S))^-$, $S'|(\text{ran } m_T(S'))^-$ are unitarily equivalent to $\bigoplus_{j=1}^{\infty} S(m_j/m_T)$, $\bigoplus_{j=1}^{\infty} S(m'_j/m_T)$ respectively, from the uniqueness assertion of Proposition A it follows that $m_j/m_T = m'_j/m_T$, $m_j = m'_j$ for each j.

The lemma is proved.

Let us put

$$t_0 = \inf\{t : \dim E((t, \|A\|])\mathfrak{R} < \infty\}. \tag{2.8}$$

Then $\sigma(A) \cap (t_0, \|A\|]$ contains only eigenvalues of finite multiplicity. Let $\{t_j\}_{j=1}^{n_A}$, $n_A = \dim E((t_0, \|A\|])\mathfrak{R} \leqq \infty$, $t_1 \geqq t_2 \geqq \ldots$, be these eigenvalues, each one being counted according its multiplicity. So we are able to state the main result of this paper:

Theorem. *The Jordan model of $S(e_A)$ is:*

(a) $\bigoplus_{j=1}^{\infty} S(e_{t_j})$ *if* $n_A = \dim E((t_0, \|A\|])\mathfrak{R} = \infty$;

(b) $\left(\bigoplus_{j=1}^{n_A} S(e_{t_j}) \right) \oplus \left(\bigoplus_{i=1}^{\infty} S(e_{t_0}) \right)$ *if* $n_A < \infty$.

Proof. We have the relation $A=A'\oplus\left(\bigoplus_{j=1}^{n_A} t_j\right)$ (here t_j is considered as a multiplication operator on a 1-dimensional Hilbert space), thus $S(e_A)=S(e_{A'})\oplus \oplus\left(\bigoplus_{j=1}^{n_A} S(e_{t_j})\right)$. If $n_A=\infty$, the conditions of Lemma 3 are satisfied for $T=S(e_{A'})$ and $S=\bigoplus_{j=1}^{\infty} S(e_{t_j})$, thus (a) follows.

Let us suppose that $n_A<\infty$. Then, if E' denotes the spectral measure of A', we have $\dim \operatorname{ran} E'((t, t_0])=\infty$ for each $t<t_0=\|A'\|$. From Lemmas 1 and 2 it follows that for each $t<t_0=\|A'\|$ the operator $S(e_{A'})|(\operatorname{ran} e_t(S(e_{A'})))^-$ is of infinite multiplicity. Let $S=S(e_{t_0})\oplus(\bigoplus_{j=1}^{\infty} S(e_{t^j}))$, $t_0\geqq t^1\geqq t^2\geqq\ldots$, be the Jordan model of $S(e_{A'})$. If $t^j=t<t_0$ for some j, it follows that $S|(\operatorname{ran} e_t(S))^-$ is of finite multiplicity, thus $S(e_{A'})|(\operatorname{ran} e_t(S(e_{A'})))^-$ is of finite multiplicity, a contradiction. It follows that $t^j=t_0$ for each j, thus $S(e_A)$ is quasisimilar to

$$\left(\bigoplus_{j=1}^{n_A} S(e_{t_j})\right)\oplus\left(\bigoplus_{i=1}^{\infty} S(e_{t_0})\right).$$

The last operator is a Jordan operator and the theorem follows from the uniqueness assertion of Proposition A.

Remark. If A acts on a finite dimensional Hilbert space we have $n_A=\operatorname{Rank} A$, $t_0=0$, and the Jordan model has the form $\bigoplus_{j=1}^{n_A} S(e_{t_j})$. Thus our theorem is verified in this case also.

Example. Let A be defined by $(Af)(x)=x\cdot f(x)$ on $\mathfrak{R}=L^2(0, 1)$. Then $\|A\|=1$ and A has no eigenvalues. It follows that the Jordan model of $S(e_A)$ is $\bigoplus_{i=1}^{\infty} S(e_1)$.

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