Acta Sci. Math., 38 (1976), 281—290

On intertwining dilations

ZOIA CEAUŞESCU

Introduction. Let T, T' be two contractions on the Hilbert space $\mathfrak{H}$ and $\mathfrak{H}'$, and U, U' their isometric dilations on $\mathfrak{K}$ and $\mathfrak{K}'$, respectively. For an operator $A \in L(\mathfrak{H}', \mathfrak{H})$ (the space of all bounded operators from $\mathfrak{H}'$ into $\mathfrak{H}$) intertwining T and T' (i.e. $TA = AT'$) let us call an *intertwining dilation* of A any operator $B \in L(\mathfrak{K}'; \mathfrak{K})$ satisfying: $P_{\mathfrak{H}} B | \mathfrak{H}' = A$, $UB = BU'$ and $B(\mathfrak{K}' \ominus \mathfrak{H}') \subset \mathfrak{K} \ominus \mathfrak{H}$. If, moreover, B satisfies $\|B\| = \|A\|$ it will be called an *exact* intertwining dilation of A. It is known that for any operator A intertwining T and T' there exists at least one exact intertwining dilation (see Th. 2. 3 of [5]).

In the present paper we are concerned with the problem of uniqueness of such an exact intertwining dilation. We reduce this problem to the similar problem for the Hahn—Banach extensions of continuous functionals on some adequate quotient spaces of projective tensor products.[1])

Our main result is contained in Section 3. Thus we show that if an operator intertwining two contractions has a unique exact intertwining dilation, then all the operators which are "dominated" (in the sense of Definition 3.1) by it have the same property (see Th. 3.2). As an illustrative example, in the last section, an application of the above theorem to Hankel operators is given.

I take this opportunity to express my gratitude to Prof. C. Foiaş, for many helpful discussions. Also I thank Prof. B. Sz.-Nagy for his useful remarks on the first version of this paper.

1. Let $\mathfrak{K}$ and $\mathfrak{G}$ be two Hilbert spaces. We shall denote by $\mathfrak{K}^* \otimes \mathfrak{G}$ the subspace of $L(\mathfrak{K}; \mathfrak{G})$ consisting of operators τ which admit a representation of the form

$$\tau = \sum_{j=1}^{n} k_j^* \otimes g_j, \quad \text{where} \quad k_j \in \mathfrak{K},\ g_j \in \mathfrak{G},\ 1 \leqq j \leqq n, \tag{1}$$

that is,

$$\tau(k) = \sum_{h=1}^{n} (k, k_j) g_j \quad (k \in \mathfrak{K}). \tag{2}$$

Received December 18, 1975, revised March 5, 1976.

[1]) This reduction already was done in some more or less particular cases (see for instance [6]).

We shall use the notation $\|\cdot\|_\pi$ for the nuclear norm on $\mathfrak{K}^*\otimes\mathfrak{G}$:

(3) $$\|\tau\| = \inf\left\{\sum_{j=1}^{n} \|k_j\|\,\|g_j\| : \tau = \sum_{j=1}^{n} k_j^*\otimes g_j\right\}.$$

The space $\mathfrak{K}^*\otimes\mathfrak{G}$ endowed with this norm will be denoted by $\mathfrak{K}^*\underset{\pi}{\otimes}\mathfrak{G}$.

An immediate result is expressed by the following

Lemma 1.1. *For a subspace $\mathfrak{H}$ of $\mathfrak{K}$ the space $\mathfrak{H}^*\underset{\pi}{\otimes}\mathfrak{G}$ can be identified with the subspace $\mathfrak{L}$ of $\mathfrak{K}^*\underset{\pi}{\otimes}\mathfrak{G}$ consisting of those $\tau\in\mathfrak{K}^*\underset{\pi}{\otimes}\mathfrak{G}$ for which*

(4) $$\tau|\mathfrak{K}\ominus\mathfrak{H} = 0.$$

On account of Lemma 1.1 we may and will identify $\mathfrak{H}^*\underset{\pi}{\otimes}\mathfrak{G}$ with the subspace $\mathfrak{L}$ defined by (4), of $\mathfrak{K}^*\underset{\pi}{\otimes}\mathfrak{G}$. We shall denote by $\mathfrak{K}^*\underset{\pi}{\hat{\otimes}}\mathfrak{G}$ and $\mathfrak{H}^*\underset{\pi}{\hat{\otimes}}\mathfrak{G}$ the completions of $\mathfrak{K}^*\underset{\pi}{\otimes}\mathfrak{G}$ and $\mathfrak{H}^*\underset{\pi}{\otimes}\mathfrak{G}$, respectively.

Let us recall some well known properties (see [7]) of the completion of projective tensor product.

(i) Every element τ of $\mathfrak{K}^*\underset{\pi}{\hat{\otimes}}\mathfrak{G}$ is the sum of an absolutely convergent series:

$$\tau = \sum_{n=0}^{\infty} k_n^*\otimes g_n, \quad \text{and} \quad \|\tau\|_\pi = \inf\left\{\sum_{n=0}^{\infty} \|k_n\|\,\|g_n\| : \tau = \sum_{n=0}^{\infty} k_n^*\otimes g_n\right\}.$$

(ii) The dual of $\mathfrak{K}^*\underset{\pi}{\hat{\otimes}}\mathfrak{G}$ is realized as the space $L(\mathfrak{G};\mathfrak{K})$.

Also, we shall consider operators U on $\mathfrak{K}$, T on $\mathfrak{H}$, and Z on $\mathfrak{G}$, and assume that $\mathfrak{H}$ is a subspace of $\mathfrak{K}$ invariant for U^*, and $U^*|\mathfrak{H}=T^*$.

We denote by $[Z, U]$ the operator on $L(\mathfrak{K};\mathfrak{G})$, defined by

(5) $$[Z, U]V = ZV - VU \quad \text{for} \quad V\in L(\mathfrak{K};\mathfrak{G}).$$

Note that $\mathfrak{K}^*\otimes\mathfrak{G}$ and $\mathfrak{H}^*\otimes\mathfrak{G}$ are invariant for $[Z, U]$, and in virtue of the condition $T^*=U^*|\mathfrak{H}$ we have

$$[Z, U]|\mathfrak{H}^*\otimes\mathfrak{G} = [Z, T]|\mathfrak{H}^*\otimes\mathfrak{G}$$

(where $[Z, T]$ is defined on $L(\mathfrak{H};\mathfrak{G})$ in the same way as $[Z, U]$ is on $L(\mathfrak{K};\mathfrak{G})$). The operators $[Z, T]$ and $[Z, U]$ can be extended continuously to $\mathfrak{H}^*\underset{\pi}{\hat{\otimes}}\mathfrak{G}$ and $\mathfrak{K}^*\underset{\pi}{\hat{\otimes}}\mathfrak{G}$, respectively. Now, denote

(6) $$\mathfrak{R}_U = ([Z, U](\mathfrak{K}^*\underset{\pi}{\hat{\otimes}}\mathfrak{G}))^-, \quad \mathfrak{R}_T = ([Z, T](\mathfrak{H}^*\underset{\pi}{\hat{\otimes}}\mathfrak{G}))^-$$

where the closures are taken in the spaces $\mathfrak{K}^*\underset{\pi}{\hat{\otimes}}\mathfrak{G}$ and $\mathfrak{H}^*\underset{\pi}{\hat{\otimes}}\mathfrak{G}$, respectively. We shall consider the quotients modulo $\mathfrak{R}_U$ and $\mathfrak{R}_T$ of the nuclear norms on $\mathfrak{K}^*\underset{\pi}{\otimes}\mathfrak{G}$ and $\mathfrak{H}^*\underset{\pi}{\hat{\otimes}}\mathfrak{G}$, respectively; thus, if ψ and φ denote the canonical epimorphism

$$\psi : \mathfrak{K}^*\underset{\pi}{\hat{\otimes}}\mathfrak{G} \to (\mathfrak{K}^*\underset{\pi}{\hat{\otimes}}\mathfrak{G})/\mathfrak{R}_U, \quad \varphi : \mathfrak{H}^*\underset{\pi}{\hat{\otimes}}\mathfrak{G} \to (\mathfrak{H}^*\underset{\pi}{\hat{\otimes}}\mathfrak{G})/\mathfrak{R}_T$$

then

$$\|\psi(\tau)\| = \inf_{\tau_1 \in \mathfrak{N}_U} \|\tau+\tau_1\|_\pi (\tau \in \mathfrak{K}^* \hat{\otimes}_\pi \mathfrak{G}) \quad \text{and} \quad \|\varphi(\tau)\| = \inf_{\tau_1 \in \mathfrak{N}_T} \|\tau+\tau_1\|_\pi (\tau \in \mathfrak{H}^* \hat{\otimes}_\pi \mathfrak{G}).$$

Since, $\mathfrak{N}_U \supset \mathfrak{N}_T$, we infer that

$$\|\psi(\tau)\| \leqq \|\varphi(\tau)\| \quad \text{for} \quad \tau \in \mathfrak{H}^* \hat{\otimes}_\pi \mathfrak{G}. \tag{7}$$

Lemma 1.2. (i) *The dual of the Banach space* $(\mathfrak{K}^* \hat{\otimes}_\pi \mathfrak{G})/\mathfrak{N}_U$ *is isometric-isomorphic to the subspace*

$$\{B \in L(\mathfrak{G}; \mathfrak{K}) : UB = BZ\} \quad of \quad L(\mathfrak{G}; \mathfrak{K}),$$

(ii) *The dual of the Banach space* $(\mathfrak{H}^* \hat{\otimes}_\pi \mathfrak{G})/\mathfrak{N}_T$ *is isometric-isomorphic to the subspace*

$$\{A \in L(\mathfrak{G}; \mathfrak{H}) : TA = AZ\} \quad of \quad L(\mathfrak{G}; \mathfrak{H}).$$

Proof. (i): Firstly, let us observe that $\{B \in L(\mathfrak{G}; \mathfrak{K})\colon UB = BZ\}$ is isometric-isomorphic to $\mathfrak{N}_U^\perp$, where we denote by $\mathfrak{N}_U^\perp$ the orthogonal of $\mathfrak{N}_U$ i.e.

$$\mathfrak{K}_U^\perp = \{f \in (\mathfrak{K}^* \hat{\otimes}_\pi \mathfrak{G})' : f | \mathfrak{N}_U = 0\}.$$

Indeed, since $L(\mathfrak{G}; \mathfrak{K})$ is isometric-isomorphic to $(\mathfrak{K}^* \hat{\otimes}_\pi \mathfrak{G})'$, for any $B \in L(\mathfrak{G}; \mathfrak{K})$ with the property $UB = BZ$ there is a unique f from $(\mathfrak{K}^* \hat{\otimes}_\pi \mathfrak{G})'$ with the properties

(a) $f(k^* \otimes g) = (Bg, k)$ $(k \in \mathfrak{K},\ g \in \mathfrak{G})$ and (b) $\|f\| = \|B\|$.

But, for this f and for any $k \in \mathfrak{K}$, $g \in \mathfrak{G}$, we also have:

$$f([Z, U](k^* \otimes g)) = (BZg, k) - (UBg, k) = 0.$$

Since the set $\{[Z, U](k^* \otimes g) : k \in \mathfrak{K},\ g \in \mathfrak{G}\}$ spans $\mathfrak{N}_U$, it results readily $f | \mathfrak{N}_U = 0$.

Conversely, since $L(\mathfrak{G}; \mathfrak{K}) \cong (\mathfrak{K}^* \hat{\otimes}_\pi \mathfrak{G})'$, for any $f \in (\mathfrak{K}^* \hat{\otimes}_\pi \mathfrak{G})'$ with $f | \mathfrak{N}_U = 0$, there exists a unique $B \in L(\mathfrak{G}; \mathfrak{K})$ satisfying conditions (a), (b) above; moreover, we have

$$((UB - BZ)g, k) = f([Z, U](k^* \otimes g)) = 0 \quad \text{for any} \quad k \in \mathfrak{K},\ g \in \mathfrak{G}.$$

Thus, the operator B has also the property $UB = BZ$.

Now, statement (i) of the Lemma results from the following general fact: If $\mathfrak{X}$ is a Banach space and $\mathfrak{Y}$ is a subspace of $\mathfrak{X}$, then the orthogonal $\mathfrak{Y}^\perp$ of $\mathfrak{Y}$ is isometric-isomorphic to the dual of the quotient space $\mathfrak{X}/\mathfrak{Y}$.

(ii): The proof is analogous to that of (i), due to the similar definition for the space $\mathfrak{H}^* \hat{\otimes}_\pi \mathfrak{G}$, and thus for $(\mathfrak{H}^* \hat{\otimes}_\pi \mathfrak{G})/R_T$ too.

Lemma 1.3. *The following two statements are equivalent:*

(P_1) *For any $A\in L(\mathfrak{G}, \mathfrak{H})$ satisfing the condition $TA=AZ$, there exists at least one exact intertwining dilation $B\in L(\mathfrak{G}; \mathfrak{K})$ of A.*

(P_2) *For any $\tau\in\mathfrak{H}^*\hat{\otimes}_\pi\mathfrak{G}$, we have $\|\psi(\tau)\|=\|\varphi(\tau)\|$.*

Proof. First, we notice that, on account of Lemma 1.2, (P_1) is equivalent to:

(P_1') *For any $f\in((\mathfrak{H}^*\hat{\otimes}_\pi\mathfrak{G})/\mathfrak{N}_T)'$ there exists an "extension" $\tilde{f}\in((\mathfrak{K}^*\hat{\otimes}_\pi\mathfrak{G})/\mathfrak{N}_U)'$ of f (i.e. $\tilde{f}\psi(\tau)=f\varphi(\tau)$ for all $\tau\in\mathfrak{H}^*\hat{\otimes}_\pi\mathfrak{G}$) such that:*

$$\|\tilde{f}\| = \|f\| \quad (or\ equivalently,\quad \|\tilde{f}\psi\| = \|f\varphi\|).$$

Indeed, if (P_1) holds then, in virtue of Lemma 1.2, for $f\in((\mathfrak{H}^*\hat{\otimes}_\pi\mathfrak{G})/\mathfrak{N}_T)'$ there is $\tilde{f}\in((\mathfrak{K}^*\hat{\otimes}_\pi\mathfrak{G})/\mathfrak{N}_U)'$ such that $\|\tilde{f}\|=\|f\|$ and $\tilde{f}\psi(h^*\otimes g)=f\varphi(h^*\otimes g)$ for all $h\in\mathfrak{H}$ and $g\in\mathfrak{G}$. Since, for $\tau\in\mathfrak{H}^*\hat{\otimes}_\pi\mathfrak{G}$ there are the representations $\tau=\sum_{n\in N}h_n^*\otimes g_n$ where the series $\sum_{n\in N}h_n^*\otimes g_n$ is absolutely convergent, and since f, $\tilde{f}$, φ, ψ, are continuous, we also have

$$f\varphi(\tau)=\tilde{f}\psi(\tau)\quad \text{for all}\quad \tau\in\mathfrak{H}^*\hat{\otimes}_\pi\mathfrak{G}.$$

The converse implication $(P_1')\Rightarrow(P_1)$ is, by Lemma 1.2, even more obvious.

Now, we assume that (P_1') holds. Let us take $\tau_0\in\mathfrak{H}^*\hat{\otimes}_\pi\mathfrak{G}$ with $\varphi(\tau_0)\neq 0$. There exists $f\in((\mathfrak{H}^*\hat{\otimes}_\pi\mathfrak{G})/\mathfrak{N}_T)'$ with the properties:

$$\|f\| = \|f\varphi\| = 1,\quad f\varphi(\tau_0) = \|\varphi(\tau_0)\|.$$

For this f there exists, according to (P_1'), $\tilde{f}\in((\mathfrak{K}^*\hat{\otimes}_\pi\mathfrak{G})/\mathfrak{N}_U)'$ such that

$$\|\tilde{f}\| = \|f\| = 1 \quad \text{and}\quad \tilde{f}\psi(\tau) = f\varphi(\tau)\quad (\tau\in\mathfrak{H}^*\hat{\otimes}_\pi\mathfrak{G}).$$

Thus, by (7),

$$\|\varphi(\tau_0)\| = \tilde{f}\psi(\tau_0) \leqq \|\tilde{f}\|\,\|\psi(\tau_0)\| = \|\psi(\tau_0)\| \leqq \|\varphi(\tau_0)\|.$$

If $\varphi(\tau_0)=0$ then, by (7), $0\leqq\|\psi(\tau_0)\|\leqq\|\varphi(\tau_0)\|=0$. Consequently, we obtain $\|\varphi(\tau)\|=\|\psi(\tau)\|$ for all $\tau\in\mathfrak{H}^*\hat{\otimes}_\pi\mathfrak{G}$.

Let us now assume that $\|\varphi(\tau)\|=\|\psi(\tau)\|$ for all $\tau\in\mathfrak{H}^*\hat{\otimes}_\pi\mathfrak{G}$. This means that the continuous canonical epimorphism

$$\varphi(\mathfrak{H}^*\hat{\otimes}_\pi\mathfrak{G}) = (\mathfrak{H}^*\hat{\otimes}_\pi\mathfrak{G})/\mathfrak{N}_T \to (\mathfrak{H}^*\hat{\otimes}_\pi\mathfrak{G})/\mathfrak{N}_U = \psi(\mathfrak{H}^*\hat{\otimes}_\pi\mathfrak{G})$$

is an isometry. Therefore, we can identify $(\mathfrak{H}^*\hat{\otimes}_\pi\mathfrak{G})/\mathfrak{N}_T$ with the subspace $(\mathfrak{H}^*\hat{\otimes}_\pi\mathfrak{G})/\mathfrak{N}_U$

of $(\mathfrak{K}^* \hat{\underset{\pi}{\otimes}} \mathfrak{G})/\mathfrak{N}_U$. Now, the implication $(P_2) \Rightarrow (P_1')$ follows from the Hahn—Banach Theorem.

It is known that if T is a contraction on $\mathfrak{H}$, U a minimal isometric dilation of T on $\mathfrak{K}$, and Z an isometry on $\mathfrak{G}$, then assertion (P_1) of Lemma 1.3 is true (cf. [5] Prop. II 2.2). Thus we have

Theorem 1.1. *Let T be a contraction on $\mathfrak{H}$, U a minimal isometric dilation of T, and Z an isometry on $\mathfrak{G}$. Then,*

$$(\mathfrak{H}^* \hat{\underset{\pi}{\otimes}} \mathfrak{G})/([Z, T](\mathfrak{H}^* \hat{\underset{\pi}{\otimes}} \mathfrak{G}))^-$$

is linear canonically isometric to the image of $\mathfrak{H}^* \hat{\underset{\pi}{\otimes}} \mathfrak{G}$ in

$$(\mathfrak{K}^* \hat{\underset{\pi}{\otimes}} \mathfrak{G})/([Z, U](\mathfrak{K}^* \hat{\underset{\pi}{\otimes}} \mathfrak{G}))^-.$$

2. In the sequel we shall only treat the case considered in Theorem 1.1; that is, T is a contraction on $\mathfrak{H}$, U is a minimal isometric dilation of T on $\mathfrak{K}$, and Z is an isometry on $\mathfrak{G}$.

Remark 2.1. Let $A \in L(\mathfrak{G}; \mathfrak{H})$ satisfy $TA = AZ$. In order that A should have a unique intertwining dilation $B \in L(\mathfrak{G}; \mathfrak{K})$ with $\|B\| = \|A\|$ it is necessary and sufficient that the functional $f \in ((\mathfrak{H}^* \hat{\underset{\pi}{\otimes}} \mathfrak{G})/\mathfrak{N}_U)'$ (where $(\mathfrak{H}^* \hat{\underset{\pi}{\otimes}} \mathfrak{G})/\mathfrak{N}_U$ is identified with $(\mathfrak{H}^* \hat{\underset{\pi}{\otimes}} \mathfrak{G})/\mathfrak{N}_T$, in virtue of Theorem 1.1), corresponding to A by: $f\psi(h^* \otimes g) = (Ag, h)$, have a unique norm-preserving extension to the space $(\mathfrak{K}^* \hat{\underset{\pi}{\otimes}} \mathfrak{G})/\mathfrak{N}_U$. On the other hand, a well-known consequence of the classical proof of the Hahn—Banach Theorem is that a functional $f \in ((\mathfrak{H}^* \hat{\underset{\pi}{\otimes}} \mathfrak{G})/\mathfrak{N}_U)'$ of norm 1 has a unique norm-preserving extension to $(\mathfrak{K}^* \hat{\underset{\pi}{\otimes}} \mathfrak{G})/\mathfrak{N}_U$ if an only if for any $\tau \notin \mathfrak{H}^* \hat{\underset{\pi}{\otimes}} \mathfrak{G}$,

$$\sup \{\operatorname{Re} f(\dot{\tau}_1) - \|\dot{\tau}_1 - \dot{\tau}\| : \dot{\tau}_1 \in (\mathfrak{H}^* \hat{\underset{\pi}{\otimes}} \mathfrak{G})/\mathfrak{N}_U\} =$$

$$= \inf \{\|\dot{\tau}_2 + \dot{\tau}\| - \operatorname{Re} f(\dot{\tau}_2) : \dot{\tau}_2 \in (\mathfrak{H}^* \hat{\underset{\pi}{\otimes}} \mathfrak{G})/\mathfrak{N}_U\}.$$

(Here, as in the sequel, we set $\dot{\tau} = \psi(\tau)$ for $\tau \in \mathfrak{K}^* \hat{\underset{\pi}{\otimes}} \mathfrak{G}$). Hence, we easily infer the following sufficient and necessary condition for that an $A \in L(\mathfrak{G}; \mathfrak{H})$, $\|A\| = 1$, satisfying $TA = AZ$ have a unique exact intertwining dilation.

For any $\varepsilon > 0$ and $\tau_0 \in (\mathfrak{K}^* \hat{\underset{\pi}{\otimes}} \mathfrak{G}) \setminus (\mathfrak{H}^* \hat{\underset{\pi}{\otimes}} \mathfrak{G})$ there exists $\tau_1, \tau_2 \in \mathfrak{H}^* \hat{\underset{\pi}{\otimes}} \mathfrak{G}$ satisfying

$$\|\dot{\tau}_1 + \dot{\tau}_2\| \leqq \|\dot{\tau}_1 - \dot{\tau}_0\| + \|\dot{\tau}_2 + \dot{\tau}_0\| < \operatorname{Re} f(\dot{\tau}_1 + \dot{\tau}_2) + \varepsilon. \tag{8}$$

3. We introduce the following definition for contractions on Hilbert spaces:

Definition 3.1. Let $A_1, A_2 \in L(\mathfrak{H}_1; \mathfrak{H}_2)$ be two contractions. We say that A_1 *Harnack-dominates* A_2 if for some positive constants C, C' we have:

$$\|D_{A_2}h\| \leqq C\|D_{A_1}h\| \quad \text{and} \quad \|(A_2-A_1)h\| \leqq C'\|D_{A_1}h\| \tag{9}$$

for all $h \in \mathfrak{H}_1$. Here D_{A_1}, D_{A_2} are the defect operators of A_1, A_2, i.e. $D_{A_i}=(1-A_i^*A_i)^{1/2}$ $(i=1, 2)$.

Remark 3.1. Let us introduce, for the contractions $A_1, A_2 \in L(\mathfrak{H}_1, \mathfrak{H}_2)$, the following isometries:

$$\hat{A}_i = \begin{pmatrix} A_i \\ D_{A_i} \end{pmatrix} : \mathfrak{H}_1 \to \begin{matrix} \mathfrak{H}_2 \\ \oplus \\ \mathfrak{D}_{A_i} \end{matrix} \quad (i=1,2),$$

where $\mathfrak{D}_{A_i}=\overline{D_{A_i}\mathfrak{H}_1}$ $(i=1, 2)$. Then, conditions (9) of Definition 3.1 are plainly equivalent to the following: There exists a bounded operator

$$K : \begin{matrix} \mathfrak{H}_2 \\ \oplus \\ \mathfrak{D}_{A_1} \end{matrix} \to \begin{matrix} \mathfrak{H}_2 \\ \oplus \\ \mathfrak{D}_{A_2} \end{matrix}$$

such that

$$K\begin{pmatrix} h_2 \\ 0 \end{pmatrix} = \begin{pmatrix} h_2 \\ 0 \end{pmatrix} \quad \text{for all} \quad h_2 \in \mathfrak{H}_2, \quad \text{and} \quad \hat{A}_2 = K\hat{A}_1. \tag{10}$$

Remark 3.2. We note that, if $\mathfrak{H}_1$ and $\mathfrak{H}_2$ coincide, then the equivalence relation for contractions on $\mathfrak{H}$, defined by: A_1 Harnack-dominates A_2, and A_2 Harnack-dominates A_1 coincides with the Harnack-equivalence as defined in [4], p. 362.

For two operators $A_1, A_2 \in L(\mathfrak{G}; \mathfrak{H})$, intertwining T and Z, denote by f_{A_1}, f_{A_2} the functionals $\in((\mathfrak{H}^* \hat{\otimes}_\pi \mathfrak{G})/\mathfrak{N}_U)'$, corresponding to A_1 and A_2, respectively, and by F_{A_1}, F_{A_2} the functionals $\in(\mathfrak{H}^* \hat{\otimes}_\pi \mathfrak{G})'$, satisfying $F_{A_1}|\mathfrak{N}_U = F_{A_2}|\mathfrak{N}_U = 0$, which correspond to f_{A_1}, f_{A_2} by virtue of the isometric-isomorphism

$$((\mathfrak{H}^* \hat{\otimes}_\pi \mathfrak{G})/\mathfrak{N}_U)' \cong \mathfrak{N}_U^\perp.$$

Lemma 3.1. *Let $A_1, A_2 \in L(\mathfrak{G}; \mathfrak{H})$ be two operators intertwining T and Z, $\|A_1\| = \|A_2\| = 1$, and such that A_1 Harnack-dominates A_2. Then,*

$$\|\tau\|_\pi - \operatorname{Re} F_{A_1}(\tau) \leqq \varepsilon \quad (\textit{for some } \varepsilon > 0 \textit{ and } \tau \in \mathfrak{H}^* \hat{\otimes}_\pi \mathfrak{G})$$

implies

$$\operatorname{Re} F_{A_1}(\tau) \leqq \operatorname{Re} F_{A_2}(\tau) + 2\varepsilon(\|K\|^2 - 1).$$

(K is the bounded operator satisfying (10), which exists by Remark 3.1.)

Proof. Let $\tau \in \mathfrak{H}^* \hat{\otimes}_\pi \mathfrak{G}$ be such that $\|\tau\|_\pi - \mathrm{Re}\, F_{A_1}(\tau) \leqq \varepsilon$ for some $\varepsilon > 0$. There exists a representation of τ, say

$$\tau = \sum_{n \in N} h_n^* \otimes g_n,$$

with

$$\|g_n\| = 1, \quad \sum_{n \in N} \|h_n\| < \infty, \quad \text{and} \quad \|\tau\|_\pi \leqq \sum_{n \in N} \|h_n\| < \|\tau\|_\pi + \varepsilon.$$

Since $F_{A_i}(h_n^* \otimes g_n) = (A_i g_n, h_n)$ $(i=1, 2)$, and since F_{A_i} are continuous it result that the series $\sum_{n \in N} (A_i g_n, h_n)$ $(i=1, 2)$ are absolutely convergent, and

$$F_{A_i}(\tau) = \sum_{n \in N} (A_i g_n, h_n)$$

Consequently,

$$\sum_{n \in N} \|h_n\| - \sum_{n \in N} \mathrm{Re}\,(A_1 g_n, h_n) \leqq 2\varepsilon.$$

Now let us notice that

$$1 - \mathrm{Re}\,(A_i g_n, f_n) = \frac{1}{2} \left\| \begin{bmatrix} A_i g_n - f_n \\ D_{A_i} g_n \end{bmatrix} \right\|^2 = \frac{1}{2} \|\hat{A}_i g_n - \hat{f}_n\|^2$$

where $f_n = \dfrac{h_n}{\|h_n\|}$ and $\hat{f}_n = \begin{pmatrix} f_n \\ 0 \end{pmatrix}$ $(n \in N)$. Since A_1 Harnack-dominates A_2 in virtue of Remark 3.1 we also have

$$\|\hat{A}_2 g_n - \hat{f}_n\|^2 = \|K(\hat{A}_1 g_n - \hat{f}_n)\|^2 \leqq \|K\|^2 \|\hat{A}_1 g_n - \hat{f}_n\|^2$$

Therefore

$$\mathrm{Re}\,(A_1 g_n, h_n) - \mathrm{Re}\,(A_2 g_n, h_n) \leqq \frac{1}{2} (\|K\|^2 - 1) \|\hat{A}_1 g_n - \hat{f}_n\|^2 \|h_n\| \quad (n \in N).$$

Whence,

$$\mathrm{Re}\, F_{A_1}(\tau) - \mathrm{Re}\, F_{A_2}(\tau) \leqq (\|K\|^2 - 1) \sum_{n \in N} \frac{1}{2} \|\hat{A}_1 g_n - \hat{f}_n\|^2 \|h_n\| =$$

$$= (\|K\|^2 - 1) \sum_{n \in N} [\|h_n\| - \mathrm{Re}\,(A_1 g_n, h_n)] < 2\varepsilon (\|K\|^2 - 1).$$

We may now state and prove our main theorem concerning the uniqueness of exact intertwining dilation.

Theorem 3.1. *Let $A_1, A_2 \in L(\mathfrak{G}; \mathfrak{H})$ be operators with the properties: $TA_1 = A_1 Z$, $TA_2 = A_2 Z$, $\|A_1\| = \|A_2\| = 1$, A_1 Harnack-dominates A_2. Then, if A_1 has a unique exact intertwining dilation so has A_2.*

Proof. By Remark 2.1, we must show that if the functional $f_{A_1} \in ((\mathfrak{H}^* \hat{\otimes}_\pi \mathfrak{G})/\mathfrak{N}_U)'$ defined by A_1 satisfies condition (8), then the functional $f_{A_2} \in ((\mathfrak{H}^* \hat{\otimes}_\pi \mathfrak{G})/\mathfrak{N}_U)'$ defined by A_2, also satisfies it.

Assume that for $\varepsilon>0$ and $\tau_0\in(\mathfrak{K}^*\hat{\otimes}_\pi\mathfrak{G})\diagdown(\mathfrak{H}^*\hat{\otimes}_\pi\mathfrak{G})$ we have

$$\|\dot{t}_1+\dot{t}_2\|\leqq\|\dot{t}_1-\dot{t}_0\|+\|\dot{t}_2+\dot{t}_0\|<\operatorname{Re}f_{A_1}(\dot{t}_1+\dot{t}_2)+\varepsilon \tag{11}$$

for some $\tau_1,\tau_2\in\mathfrak{H}^*\hat{\otimes}_\pi\mathfrak{G}$. Since $\|\dot{t}\|=\|\varphi(\tau)\|=\|\psi(\tau)\|$ for all $\tau\in\mathfrak{H}^*\hat{\otimes}_\pi\mathfrak{G}$, there exists $\tau'\in\mathfrak{N}_T$ such that

$$\|\tau_1+\tau_2+\tau'\|_\pi<\|\varphi(\tau_1+\tau_2)\|+\varepsilon=\|\dot{t}_1+\dot{t}_2\|+\varepsilon.$$

Denote $\tau_2'=\tau_2+\tau'$ and note that

$$\|\dot{t}_1+\dot{t}_2'\|=\|\dot{t}_1+\dot{t}_2\| \quad\text{and}\quad f_{A_i}(\dot{t}_1+\dot{t}_2')=f_{A_i}(\dot{t}_1+\dot{t}_2).$$

Then, from (11) we readily infer that

$$\|\tau_1+\tau_2'\|_\pi<\operatorname{Re}f_{A_1}(\dot{t}_1+\dot{t}_2')+2\varepsilon=\operatorname{Re}F_{A_1}(\tau_1+\tau_2')+2\varepsilon.$$

Consequently, in virtue of Lemma 3.1, it follows

$$\operatorname{Re}F_{A_1}(\tau_1+\tau_2')\leqq\operatorname{Re}F_{A_2}(\tau_1+\tau_2')+2\varepsilon(\|K\|^2-1)$$

or, equivalently,

$$\operatorname{Re}f_{A_1}(\dot{t}_1+\dot{t}_2)\leqq\operatorname{Re}f_{A_2}(\dot{t}_1+\dot{t}_2)+2\varepsilon(\|K\|^2-1).$$

Whence it results that f_{A_2} satisfies the condition

$$\|\dot{t}_1-\dot{t}_0\|+\|\dot{t}_2+\dot{t}_0\|<\operatorname{Re}f_{A_2}(\dot{t}_1+\dot{t}_2)+2\varepsilon(\|K\|^2-1).$$

Thus, we can conclude that f_{A_2} satisfies (8) too.

As a corollary of the previous theorem we have the following more general result:

Theorem 3.2. *Let T, T' be two contractions on the Hilberts spaces $\mathfrak{H}$ and $\mathfrak{H}'$, respectively. Moreover let $A_1, A_2\in L(\mathfrak{H}';\mathfrak{H})$ satisfy the conditions:*

$TA_1=A_1T'$, $TA_2=A_2T'$, $\|A_1\|=\|A_2\|=1$, A_1 Harnack-dominates A_2 Then, if A_1 has a unique exact intertwining dilations so has A_2.

Indeed, denoting by Z the minimal isometric dilation of T' it is known (see [5], Th. 2.3). that all exact intertwining dilations of A_i $(i=1,2)$ are obtained as exact intertwining dilations of the operators $B_i=A_iP_{\mathfrak{H}'}$ $(i=1,2)$ intertwining T and Z.

4. Let T, T' be two contractions on the Hilbert space $\mathfrak{H}$ and $\mathfrak{H}'$, and let U, U' be their minimal isometric dilations on the spaces $\mathfrak{K}$ and $\mathfrak{K}'$, respectively.

Theorem 4.1. *Let $B_1, B_2\in L(\mathfrak{K}';\mathfrak{K})$ have the properties: $\|B_1\|=\|B_2\|=1$, $UB_i=B_iU'$, $PB_i(I-P')=0$ $(i=1,2)$* where, $P=P_{\mathfrak{H}}$, $P'=P_{\mathfrak{H}'}$ *B_1 Harnack-dominates B_2, and let $A_1, A_2\in L(\mathfrak{H}';\mathfrak{H})$ be the operators $A_i=PB_i|\mathfrak{H}'$ $(i=1,2)$. Then, if B_1 is an exact intertwining dilation of A_1, then A_2 is an exact intertwining dilation of A_2; moreover, if B_1 is the unique exact intertwining dilation for A_1, so is B_2 for A_2.*

Proof. First, by hypothesis we observe that $PB_i=A_iP'$ and A_i is intertwining T and T'. Thus, B_i is an intertwining dilation of A_i $(i=1,2)$.

Now, in order to prove that B_2 is an exact intertwining dilation for A_2 if B_1 is so for A_1, it suffices to show that $\|A_2\|=1$.

Clearly, we have (by definition of A_2) $\|A_2\|\leqq 1$.

For the converse inequality we observe that, since B_1 Harnack-dominates B_2, i.e. $\|D_{B_2}k'\|\leqq C\|D_{B_1}k'\|$ and $\|(B_2-B_1)k'\|\leqq C'\|D_{B_1}k'\|$ with $C, C'>0$, we have for $h'\in\mathfrak{H}'$

$$\|(1-P)B_2h'\| \leqq \|(1-P)B_1h'\|+\|(1-P)(B_2-B_1)h'\| \leqq \|D_{A_1}h'\|+\|(B_2-B_1)h'\| \leqq$$
$$\leqq \|D_{A_1}h'\|+C'\|D_{B_1}h'\| \leqq (1+C')\|D_{A_1}h'\|$$

and therefore,

$$\|D_{A_2}h'\|^2 = \|D_{B_2}h'\|^2+\|(1-P)B_2h'\|^2 \leqq \big(C^2+(1+C')^2\big)\|D_{A_1}h'\|^2 = C''\|D_{A_1}h'\|^2,$$

for any $h'\in\mathfrak{H}$.

Since $\|A_1\|=1$, we infer from this inequality that $\|A_2\|=1$ too, thus B_2 is an exact intertwining dilation of A_2.

The above relation with the following one:

$$\|(A_2-A_1)h'\| \leqq \|(B_2-B_1)h'\| \leqq C'\|D_{B_1}h'\| \leqq C'\|D_{A_1}h'\| \quad (h'\in\mathfrak{H}')$$

means that A_1 Harnack-dominates A_2. Now the second statement of this theorem can be obtained by referring to Theorem 3.2.

Lemma 4.1. *Let $B_1, B_2\in L(\mathfrak{K}';\mathfrak{K})$, $\|B_1\|=\|B_2\|=1$ be of the form $B_i=B_0\oplus S_i$ where S_i are strict contractions $(i=1, 2)$. Then B_1, B_2 Harnack-dominate each other.*

Proof. Consider the decomposition $\mathfrak{K}'=\mathfrak{K}_0'\oplus\mathfrak{K}_1'$ for which

$$B_1P_{\mathfrak{K}_0'} = B_2P_{\mathfrak{K}_0'} = B_0 \quad\text{and}\quad S_i = B_iP_{\mathfrak{K}_1'} = B_i(1-P_{\mathfrak{K}_0'})$$

and note that

$$\|D_{B_i}k'\|^2 = (\|k_0'\|^2-\|B_0k_0'\|^2)+(\|k_1'\|^2-\|S_ik_1'\|^2) \geqq$$
$$\geqq \|k_1'\|^2-\|S_ik_1'\|^2 \geqq (1-\|S_i\|^2)\|k_1'\|^2, \quad\text{where}\quad k_0' = P_{\mathfrak{K}_0'}k', \quad k_1' = P_{\mathfrak{K}'}k'.$$

Whence, by taking $C=\max\{(1-\|S_1\|^2)^{-1/2}, (1-\|S_2\|^2)^{-1/2}\}$ it follows

$$\|P_{K_1'}k'\| \leqq C\|D_{B_i}k'\| \quad\text{for all}\quad k'\in\mathfrak{K}'.$$

Therefore, we have $\|(B_2-B_1)k'\|\leqq\|S_2-S_1\|\|k_1'\|\leqq C'\|D_{B_i}k'\|$ and also

$$\|D_{B_2}k'\|^2 = \|k'\|^2-\|B_0k_0'\|^2-\|S_2k_1'\|^2 = \|D_{B_1}k'\|^2+(\|S_1k_1'\|-\|S_1k_1'\|)(\|S_2k_1'\|+\|S_2k_1'\|)$$
$$\leqq \|D_{B_1}k'\|^2+\|S_1-S_2\|(\|S_1\|+\|S_2\|)\|k_1'\|^2;$$

hence $\|D_{B_2}k'\|\leqq C''\|D_{B_1}k'\|$ for all $k'\in\mathfrak{K}$, where C', C'' are constants.

Thus B_1 Harnack-dominates B_2. By symmetry B_2 also Harnack-dominates B_1.

Theorem 4.1 and Lemma 4.1 have the following

Corollary 4.1. *Let $B_1, B_2\in L(\mathfrak{K}';\mathfrak{K})$ be two operators as in Lemma 4.1, intertwining U and U' and such that: $B_i(\mathfrak{K}'\ominus\mathfrak{H}')\subset\mathfrak{K}\ominus\mathfrak{H}$ $(i=1, 2)$. Then, B_1 is an* *exact intertwining dilation of* $A_1 = P_{\mathfrak{H}} B_1 | \mathfrak{H}'$, *if and only if* B_2 *is an exact intertwining dilation of* $A_2 = P_{\mathfrak{H}} B_2 | \mathfrak{H}'$; *moreover,* B_1 *is the unique exact intertwining dilation for* A_1 *if and only if* B_2 *is so for* A_2.

In virtue of Theorems 2 and 5 of [2], we also have the following corollary of Theorem 4.1, concerning the Hankel operators.[2])

Corollary 4.2. *Let* $F_1, F_2 \in L^\infty(\mathfrak{E}, \mathfrak{F})$ ($\mathfrak{E}, \mathfrak{F}$-separable Hilbert spaces) *have the properties:*

$\|F_1\| = \|F_2\| = 1$,

$F_1(t) = F_2(t)$ *whenever* $\max\{\|F_1(t)\|, \|F_2(t)\|\} > 1 - \theta$ *for some fixed* θ, $0 < \theta < 1$.

Then, if one of these functions is a minifunction for its Hankel operator, then so is the other. Moreover, if one of them is the unique minifunction of its Hankel operator so is the other.

Proof. Set $\sigma = \{t \in [0, 1]: \max\{\|F_1(t)\|, \|F_2(t)\|\} > 1 - \theta\}$, and $\mathfrak{L}_0 = \chi_\sigma L^2(\mathfrak{E})$, $\mathfrak{L}_1 = \chi_{[0,1] \setminus \sigma} L^2(\mathfrak{E})$ where χ_σ is the characteristic function of σ. Then $L^2(\mathfrak{E}) = \mathfrak{L}_0 \oplus \mathfrak{L}_1$. Also, denoting by B_i the operators: $f \to F_i f$ from $L^2(\mathfrak{E})$ to $L^2(\mathfrak{F})$ $(i = 1, 2)$, we observe that

$$B_1 P_{\mathfrak{L}_0} = B_2 P_{\mathfrak{L}_0}, \quad B_i \mathfrak{L}_0 \subset \chi_\sigma L^2(\mathfrak{F}) \quad \text{and} \quad B_i \mathfrak{L}_1 \subset \chi_{[0,1] \setminus \sigma} L^2(\mathfrak{F}).$$

Thus the operators B_i can be written $B_i = B_0 \oplus S_i$ where

$$B_0 = B_i P_{\mathfrak{L}_0}, \quad S_i = B_i P_{\mathfrak{L}_1} \quad \text{and} \quad \|S_i\| < 1 \quad (i = 1, 2).$$

Now Corollary 4.2 follows at once by Corollary 4.1.

References

[1] V. M. Adamyan, D. Z. Arov, M. G. Krein, Бесконечные блочно-Ганкелевы матрицы и связанные с ними проблемы продолжения, *Изв. Акад. Наук Армянской ССР*, **6** (1971), 87—112.

[2] L. B. Page, Bounded and compact vectorial Hankel operators, *Trans. Amer. Math. Soc.*, **150** (1970), 529—539.

[3] I. Suciu, *Function algebras*, Ed. Acad. and Nordhoff Inter. Publ. (Bucureşti and Leyden, 1975).

[4] I. Suciu, Analytic relations between functional models for contractions, *Acta Sci. Math.*, **34** (1973), 359—365.

[5] B. Sz.-Nagy, C. Foiaş, *Harmonic analysis of operators on Hilbert space*, Akadémiai Kiadó and North-Holland Publ. Comp. (Budapest and Amsterdam/London, 1970).

[6] B. Sz.-Nagy, C. Foiaş, Commutants de certains opérateurs, *Acta Sci. Math.*, **29** (1968), 1—17.

[7] F. Trèves, *Topological vector spaces, distributions and kernels*, Acad. Press (New York/London, 1967).

„INCREST"
CALEA VICTORIEI 114
BUCHAREST 22, ROMANIA

[2]) This corollary can be also obtained as a consequence of Theorems 1.3 and 3.1 of [1].