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Universal quasinilpotent operators

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1. Introduction. Let $\mathfrak{H}$ be a complex Hilbert space of (topological) dimension h and let $\mathcal{L}(\mathfrak{H})$ be the algebra of all (bounded linear) operators in $\mathfrak{H}$. Given T in $\mathcal{L}(\mathfrak{H})$, let $\mathcal{S}(T)=\{WTW^{-1}$: W is invertible in $\mathcal{L}(\mathfrak{H})\}$ ("similarity orbit" of T). What is $\mathcal{S}(T)^-$, the norm-closure of $\mathcal{S}(T)$? In this note it will be shown that the similarity orbit of a quasinilpotent perator could be surprisingly large. The norm-closure of the set $\mathcal{N}(\mathfrak{H})=\{Q\in\mathcal{L}(\mathfrak{H})$: Q is nilpotent$\}$ was completely characterized in [1] (separable case) and [11] (non-separable case); it was shown, in particular, that every quasinilpotent operator belongs to $\mathcal{N}(\mathfrak{H})^-$. Since $\mathcal{N}(\mathfrak{H})^-$ is invariant under similarities, it readily follows that $\mathcal{S}(Q)^-$ must be contained in $\mathcal{N}(\mathfrak{H})^-$ for every quasinilpotent operator Q. The main result says that the converse inclusion is also true for a suitably chosen Q.

First of all, consider the finite dimensional case. Assume that T is a nilpotent operator on a Hilbert space $\mathfrak{H}$ of dimension n $(0<n<\infty)$. Then there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ with respect to which T can be written as a matrix $T=(t_{jk})_{j,k=1}^n$, where $t_{jk}=0$ for all $j\geq k$ (i.e., an upper triangular matrix with 0's in the diagonal). Given $\varepsilon>0$, let $T_\varepsilon=(t_{jk,\varepsilon})_{j,k=1}^n$, where $t_{jk,\varepsilon}=t_{jk}$ if $k\neq j+1$ or $t_{j,j+1}\neq 0$ and $t_{j,j+1,\varepsilon}=\varepsilon$ if $k=j+1$ and $t_{j,j+1}=0$. Clearly, $\|T-T_\varepsilon\|\leq\varepsilon$ and T_ε is similar to its Jordan form, given by the matrix $Q_{un}=(\delta_{j+1,k})$, where δ_{jk} denotes the Kronecker delta. Since ε can be chosen arbitrarily small, we have arrived to the following result:

Lemma 1. *Let $\mathfrak{H}$ be an n-dimensional Hilbert space* $(0<n<\infty)$ *and let* $Q_{un}=(\delta_{j+1,k})$ *(with respect to some ONB). Then $\mathcal{S}(Q_{un})^-$ coincides with the set of all nilpotent operators in $\mathfrak{H}$.*

2. The ideal of compact operators. Let $\mathcal{K}(\mathfrak{H})$ denote the ideal of compact operators on a Hilbert space $\mathfrak{H}$ of infinite dimension h.

Lemma 2. *The compact quasinilpotent operator* $K_{uh}\approx\left(\bigoplus_{n=1}^{\infty} 1/nQ_{un}\right)\oplus 0$, *where* 0 *is the zero operator acting on a subspace of dimension* h ($\approx$ *means "unitarily equivalent to") has the property:* $\mathcal{S}(K_{uh})^- = \{K\in\mathcal{K}(\mathfrak{H})$: K is quasinilpotent$\}$.

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Proof. Let K be a compact quasinilpotent operator. Then $\mathfrak{H}=\mathfrak{H}_0\oplus\mathfrak{H}_1$, where $\mathfrak{H}_0, \mathfrak{H}_1$ reduce K, $\dim\mathfrak{H}_0=\aleph_0$ and $K|\mathfrak{H}_1=0$ (the vertical bar denotes restriction). Now it is clear that, by a trivial modification of the proof given by R. G. Douglas in [8] for the case when $\mathfrak{H}$ is separable, it can be shown that K is a norm limit of finite rank nilpotents. On the other hand, we already know that the set of all compact quasinilpotents is closed in $\mathcal{L}(\mathfrak{H})$ (see, e.g., [12]). Thus, in order to complete the proof we only have to show that $\mathcal{S}(K_{uh})^-$ actually contains every finite rank nilpotent.

Let F be a finite rank nilpotent in $\mathcal{L}(\mathfrak{H})$. Then there exists a finite dimensional subspace $\mathfrak{H}_n$ of dimension n, $0<n<\infty$, reducing F such that $F|\mathfrak{H}_n^\perp=0$. Up to a unitary transformation (of $\mathfrak{H}$ onto itself) we can obviously assume that $\mathfrak{H}_n$ is the space of Q_{un}. Hence, $F|\mathfrak{H}_n\in\mathcal{S}(Q_{un})^-$ (use Lemma 1).

Since $K_{uh}=(1/n)Q_{un}\oplus K_n''$ (with respect to the decomposition $\mathfrak{H}=\mathfrak{H}_n\oplus\mathfrak{H}_n^\perp$), where K_n'' is a quasinilpotent operator acting on $\mathfrak{H}_n^\perp$, it follows from [16] that $(1/n)Q_n\oplus 0\in\mathcal{S}(K_{uh})^-$. Since Q_n and $(1/n)Q_n$ are similar, we conclude that $F\in\mathcal{S}(K_{uh})^-$. □

This result suggests the following

Definition 1. A (necessarily quasinilpotent, but not nilpotent) operator $Q_u(\mathcal{J})$ satisfying the equality $\mathcal{S}[Q_u(\mathcal{J})]^-=\{Q\in\mathcal{J}: Q \text{ is quasinilpotent}\}$ for a given closed bilateral ideal $\mathcal{J}$ of $\mathcal{L}(\mathfrak{H})$ will be called a *universal quasinilpotent for the ideal* $\mathcal{J}$.

Let K be an arbitrary compact quasinilpotent, but not nilpotent, operator. Then ([8]) there exists a vector $x\in\mathfrak{H}$ such that $K^n x\neq 0$ for all $n=0, 1, 2, \ldots$. Let $\mathfrak{H}$ be the (closed) subspace spanned by $\{K^n x\}_{n=0}^\infty$ and let

$$K=\begin{bmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{bmatrix}$$

be the matrix representation of K with respect to the orthogonal decomposition $\mathfrak{H}=\mathfrak{H}_x\oplus\mathfrak{H}_x^\perp$. Clearly, K_{11} and K_{22} are quasinilpotent operators, so that we can proceed as in [12] in order to show that $K_{11}\oplus 0\in\mathcal{S}(K)^-$. Assuming that K_{11} is similar to a compact weighted shift with non-zero weights, it is not difficult to prove (by using the arguments of [12] and the proof of Lemma 2) that K_{11} and, a fortiori, K are compact universal quasinilpotents. This suggests the following

Conjecture 1. *A compact quasinilpotent operator is either nilpotent or a compact universal quasinilpotent.*

The above observations reduce this conjecture to the analysis of those compact quasinilpotents having a cyclic vector.

3. Similarity orbits of certain normal operators. Our next step will be a partial characterization of the set $\mathscr{S}(N)^-$ for the case when N is a normal operator. (A more complete description of this case will be given in an oncoming article [13].)

The closed bilateral ideals of $\mathscr{L}(\mathfrak{H})$ have been completely characterized by several authors ([3; 6; 14]): Let α be a cardinal number such that $\aleph_0 \leqq \alpha \leqq h = \dim \mathfrak{H}$ and let $\mathscr{J}_\alpha$ be the norm-closure of the set of all operators T in $\mathscr{L}(\mathfrak{H})$ such that $\dim (T\mathfrak{H})^- < \alpha$. Then $\mathscr{J}_\alpha$ is a closed bilateral ideal of $\mathscr{L}(\mathfrak{H})$ and every such proper (non-zero) ideal has this form. The *weighted spectrum* of $A \in \mathscr{L}(\mathfrak{H})$ corresponding to $\mathscr{J}_\alpha$ is the spectrum $\Lambda_\alpha(A)$ of the canonical projection of A in the quotient algebra $\mathscr{L}(\mathfrak{H})/\mathscr{J}_\alpha$; namely, $\Lambda_{\aleph_0}(A) = E(A)$ is the usual Calkin essential spectrum of A, and $\Lambda_h(A)$ is the *heavy spectrum* (i.e., the one corresponding to the largest ideal). For the analysis of these weighted spectra, as well as for the definition and properties of the *approximate nullity* $\delta(A)$ of an operator A, the reader is referred to [4; 11]. We recall that, in the separable case, the condition $\delta(\lambda - A) = \delta(\bar{\lambda} - A^*)$ (where A^* denotes the adjoint of the operator A) for all complex λ is equivalent to saying that if $(\lambda - A)$ is a semi-Fredholm operator, then its index is 0, i.e., A is a bi-quasitriangular operator in the sense of [1; 2].

Theorem 1. *Let N be a normal operator such that $\Lambda(N)$ (the spectrum of N) is a perfect set and coincides with $\Lambda_h(N)$. Then $\mathscr{S}(N)^-$ contains every operator $A \in \mathscr{L}(\mathfrak{H})$ such that $\Lambda(A) = \Lambda_h(A) = \Lambda(N)$ and $\delta(\lambda - A) = \delta(\bar{\lambda} - A^*)$* for all complex λ.

Let A be as in Theorem 1. By using the results of [2, Theorem 2.2] and [11] we can see that, given $\varepsilon > 0$, there exists an operator A' satisfying the same hypotheses as A such that $\|A - A'\| < \varepsilon$ and

$$A' \approx \begin{bmatrix} N & 0 & T_1 \\ 0 & N & L_1 \\ 0 & 0 & L_2 \end{bmatrix} = \begin{bmatrix} N & T \\ 0 & L \end{bmatrix}, \quad T = [0 \ T_1], \quad L = \begin{bmatrix} N & L_1 \\ 0 & L_2 \end{bmatrix}.$$

(All these matrices of operators are referred to suitable orthogonal direct sum decompositions of the underlying spaces.) It readily follows that L also satisfies the hypotheses of Theorem 1. Therefore, by [11; 18], L is a norm limit of algebraic operators with spectra contained in $\Lambda(N)$; furthermore, by an easy approximation argument, L can be actually approximated in the norm by operators which are similar to normal operators with *finite spectrum* contained in $\Lambda(N)$. Thus, in order to complete the proof of Theorem *1* it will be enough to prove the following weaker version of it:

Theorem 1'. *Let N be a normal operator in $\mathscr{L}(\mathfrak{H})$ such that $\Lambda(N) = \Lambda_h(N)$ is a perfect set, let T: $\mathfrak{H}' \to \mathfrak{H}$ be an arbitrary continuous linear mapping from a Hilbert space $\mathfrak{H}'$, $\dim \mathfrak{H}' = h' \leqq h$, and let $M, W \in \mathscr{L}(\mathfrak{H}')$, where M is normal with a finite* *spectrum contained in $\Lambda(N)$ and W is invertible. Then $\mathcal{S}(N)^-$ contains every operator in $\mathcal{L}(\mathfrak{H})$ unitarily equivalent to*

$$\begin{bmatrix} N & T \\ 0 & WMW^{-1} \end{bmatrix}$$

(with respect to the orthogonal direct sum decomposition $\mathfrak{H}\oplus\mathfrak{H}'$).

The proof will be given in a series of lemmas.

Lemma 3. *Let N be as in Theorem* 1 *and let $\lambda\in\Lambda(N)$. If*

$$A \approx \begin{bmatrix} N & T \\ 0 & \lambda I' \end{bmatrix}$$

(I' = identity on $\mathfrak{H}'$), then $A\in\mathcal{S}(N)^-$.

Proof. Clearly, we can translate N by a multiple of the identity and assume that $\lambda=0$. According to the characterization of the norm closure of $\mathcal{U}(N)=\{UNU^{-1}\colon U \text{ is unitary}\}$ given in [12] (see also [7]), $\mathcal{U}(N)^-$ (which is obviously contained in $\mathcal{S}(N)^-$) contains every normal operator $N'\approx N\oplus 0'$, where $0'$ denotes the zero operator in $\mathcal{U}'$.

Case I: h' is finite.

In this case A is a compact perturbation of an operator N' as above and the result follows from [10, Lemma 1].

Case II: $\aleph_0\leqq h'<h$.

Proceeding as in [11], it is possible to find an orthogonal direct sum decomposition $\mathfrak{H}=\mathfrak{H}_0\oplus\mathfrak{H}''$, such that $\dim\mathfrak{H}''=h'$, $\dim\mathfrak{H}_0=h$ and

$$\begin{bmatrix} N & T \\ 0 & 0' \end{bmatrix} = N_0\oplus\begin{bmatrix} N'' & T'' \\ 0'' & 0' \end{bmatrix}$$

with respect to $\mathfrak{H}_0\oplus\mathfrak{H}''\oplus\mathfrak{H}'$, where $N_0\in\mathcal{L}(\mathfrak{H}_0)$, $N''\in\mathcal{L}(\mathfrak{H}'')$ are normal operators satisfying $\Lambda(N_0)=\Lambda_h(N_0)=\Lambda(N'')=\Lambda_{h'}(N'')=\Lambda(N)$.

This reduces our problem to

Case III: $h'=h$.

Given $\varepsilon>0$, we can find an ε', $0<\varepsilon'<\min\{\varepsilon, 1\}$ such that if $\Delta_0=\{\lambda\colon |\lambda|\geqq\varepsilon'\}$ and $\Delta_0'=\{\lambda\colon|\lambda|<\varepsilon\}$, then $\Delta_0\cap\Lambda(N)$ and $[\Delta_0'\cap\Lambda(N)]^-$ are nonempty perfect sets. To simplify the notation, we can directly assume that $\varepsilon'=\varepsilon$ and $0<\varepsilon<1$. Let $E(\cdot)$ be the spectral measure of N; then $E(\Delta_0)\mathfrak{H}=\mathfrak{H}_0$ and $E(\Delta_0')\mathfrak{H}=\mathfrak{H}_0'$ are complementary h-dimensional orthogonal reducing subspaces of N and N can be written as $N=$

$=N_0 \oplus N_0'$, where $N_0 \in \mathscr{L}(\mathfrak{H}_0)$ and $N_0' \in \mathscr{L}(\mathfrak{H}_0')$, with respect to this decomposition. Then we can also write

$$B = \begin{bmatrix} N & T \\ 0 & 0' \end{bmatrix} = \begin{bmatrix} N_0 & 0 & T_1 \\ 0 & N_0' & T_2 \\ 0 & 0 & 0' \end{bmatrix}$$

with respect to $\mathfrak{H}_0 \oplus \mathfrak{H}_0' \oplus \mathfrak{H}'$.

Combining T_2 with an isometry V from $\mathfrak{H}_0'$ onto $\mathfrak{H}'$ and using the polar decomposition of VT_2, it is not difficult to see that $\mathfrak{H}_0'$ and $\mathfrak{H}'$ can be written as orthogonal direct sums $\mathfrak{H}_0' = \mathfrak{H}_{0a}' \oplus \mathfrak{H}_{0b}'$ and $\mathfrak{H}' = \mathfrak{H}_a' \oplus \mathfrak{H}_b'$, where $\dim \mathfrak{H}_{0a}' = \dim \mathfrak{H}_{0b}' = \dim \mathfrak{H}_a' = \dim \mathfrak{H}_b' = h$ and $T_2 \mathfrak{H}_a' \subset \mathfrak{H}_{0a}'$ and $T_2 \mathfrak{H}_b' \subset \mathfrak{H}_{0b}'$. Therefore, we can write $T_2 = T_{2a} \oplus T_{2b}$, where $T_{2a}(T_{2b}) = T_2 | \mathfrak{H}_a'$ ($\mathfrak{H}_b'$, resp.) and

$$B = \begin{bmatrix} N_0 & 0 & T_1 \\ 0 & N_0' & T_{2a} \oplus T_{2b} \\ 0 & 0 & 0 \oplus 0 \end{bmatrix}.$$

Let $\Delta_j = \{\lambda : \varepsilon_{j+1} \leqq |\lambda| < \varepsilon_j\}$, $j = 1, 2, 3, 4$, be such that $[\Delta_j \cap \Lambda(N_0')]^-$ is perfect for all j and $0 = \varepsilon_5 < \varepsilon_4 < \varepsilon_3 < \varepsilon_2 < \varepsilon^2 < \varepsilon_1 = \varepsilon$. Proceeding as in the first part of the proof, we van decompose $\mathfrak{H}_0' = \bigoplus_{j=1}^{4} \mathfrak{H}_j$ and $N_0' = \bigoplus_{j=1}^{4} N_j$ in such a way that $N_j \in \mathscr{L}(\mathfrak{H}_j)$ and $\Lambda(N_j) = [\Delta_j \cap \Lambda(N_0')]^-$. Now choose arbitrary normal operators $M_1 \in \mathscr{L}(\mathfrak{H}_{0a}')$, $M_2 \in \mathscr{L}(\mathfrak{H}_{0b}')$, $M_3 \in \mathscr{L}(\mathfrak{H}_a')$ and $M_4 \in \mathscr{L}(\mathfrak{H}_b')$ such that $M_j \approx N_j$, $j = 1, 2, 3, 4$. Since $\Lambda(M_1) \cap \Lambda(M_3) = \Lambda(M_2) \cap \Lambda(M_4) = \emptyset$, it follows from ROSENBUML's Corollary ([15, Corollary 0.15]) that the operators $M_1 \oplus M_3$ and $M_2 \oplus M_4$ are similar to

$$\begin{bmatrix} M_1 & T_{2a} \\ 0 & M_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} M_2 & T_{2b} \\ 0 & M_4 \end{bmatrix},$$

respectively. Hence,

$$R = N_0 \oplus \begin{bmatrix} M_1 & T_{2a} \\ 0 & M_3 \end{bmatrix} \oplus \begin{bmatrix} M_2 & T_{2b} \\ 0 & M_4 \end{bmatrix} = \begin{bmatrix} N_0 & 0 & 0 \\ 0 & M_1 \oplus M_2 & T_{2a} \oplus T_{2b} \\ 0 & 0 & M_3 \oplus M_4 \end{bmatrix} =$$

$$= \begin{bmatrix} N_0 & 0 & 0 \\ 0 & M_1 \oplus M_2 & T_2 \\ 0 & 0 & M_3 \oplus M_4 \end{bmatrix}$$

is similar to N. Thus, if $X = -N_0^{-1} T_1$ and

$$W = \begin{bmatrix} I & 0 & X \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad \text{then} \quad WRW^{-1} = \begin{bmatrix} N_0 & 0 & T_1 - X(M_3 \oplus M_4) \\ 0 & M_1 \oplus M_2 & T_2 \\ 0 & 0 & M_3 \oplus M_4 \end{bmatrix}.$$

Since $\|B-WRW^{-1}\| \leqq \|X(M_3\oplus M_4)\| + \|N_0'-M_1\oplus M_2\| + \|M_3\oplus M_4\| \leqq \varepsilon^2\|N_0^{-1}\| \cdot \|T\| + 2\varepsilon+\varepsilon^2 \leqq \varepsilon\|T\| + 2\varepsilon+\varepsilon^2 < (3+\|T\|)\varepsilon$ and WRW^{-1} is similar to N, we conclude that dist $[A, \mathscr{S}(N)] < (3+\|T\|)\varepsilon$, whence the result follows. □

Lemma 4. *Lemma 3 remains true if N is replaced by WNW^{-1}, for some invertible W.*

Proof. Clearly, $\mathscr{S}(N)^- = \mathscr{S}(WNW^{-1})^-$ and therefore it is enough to show that if

$$A \approx \begin{bmatrix} WNW^{-1} & T \\ 0 & \lambda I' \end{bmatrix},$$

then $A \in \mathscr{S}(N)^-$.

By Lemma 3, every operator $A' \in \mathscr{L}(\mathfrak{H})$ such that

$$A' \approx \begin{bmatrix} N & W^{-1}T \\ 0 & \lambda I' \end{bmatrix}$$

belongs to $\mathscr{S}(N)^-$.

On the other hand,

$$\begin{bmatrix} W & 0 \\ 0 & I' \end{bmatrix} \begin{bmatrix} N & W^{-1}T \\ 0 & \lambda I' \end{bmatrix} \begin{bmatrix} W & 0 \\ 0 & I' \end{bmatrix}^{-1} = \begin{bmatrix} WNW^{-1} & T \\ 0 & \lambda I' \end{bmatrix}.$$

Since $\mathscr{S}(N)^-$ is invariant under similarities ([12]), it readily follows that $A \in \mathscr{S}(N)^-$. □

Lemma 5. *Let N be as in Theorem 1, let $\{\lambda_1, \ldots, \lambda_m\}$ be a finite subset of $\Lambda(N)$, let I_j be the identity operator on a Hilbert space $\mathfrak{H}_j$ of dimension $h_j \leqq h$, and let $M = \bigoplus_{j=1}^{m} \lambda_j I_j \in \mathscr{L}(\mathfrak{H}')$, where $\mathfrak{H}' = \bigoplus_{j=1}^{m} \mathfrak{H}_j$. Then $\mathscr{S}(N)^-$ contains every operator $A \in \mathscr{L}(\mathfrak{H})$ unitarily equivalent to*

$$\begin{bmatrix} N & T \\ 0 & M \end{bmatrix}.$$

(With respect to the orthogonal direct sum $\mathfrak{H} \oplus \mathfrak{H}'$.)

Proof. This follows by induction over m. For $m=1$, it is the result of Lemma 3. Assume that the result is true for $m=n$ and let $m=n+1$. Set $M = M_n \oplus \lambda_{n+1} I_{n+1}$, where $M_n = \bigoplus_{j=1}^{n} \lambda_j I_j$; then

$$\begin{bmatrix} N & T \\ 0 & M \end{bmatrix} = \begin{bmatrix} N & T_n & T_{n+1} \\ 0 & M_n & 0 \\ 0 & 0 & \lambda_{n+1} I_{n+1} \end{bmatrix} = \begin{bmatrix} N_n & T_{n+1} \\ 0 & \lambda_{n+1} I_{n+1} \end{bmatrix}, \quad \text{where} \quad N_n = \begin{bmatrix} N & T_n \\ 0 & M_n \end{bmatrix}.$$

(The first matrix corresponds to the decomposition $\mathfrak{H}\oplus\mathfrak{H}'$, the second one to $\mathfrak{H}\oplus\oplus\left(\bigoplus_{j=1}^{n}\mathfrak{H}_j\right)\oplus\mathfrak{H}_{n+1}$ and the third one to $\left[\mathfrak{H}\oplus\left(\bigoplus_{j=1}^{n}\mathfrak{H}_j\right)\right]\oplus\mathfrak{H}_{n+1}$; the matrix of N_n corresponds to the decomposition $\mathfrak{H}\otimes\left(\bigoplus_{j=1}^{n}\mathfrak{H}_j\right)\Big)$.

By our inductive hypothesis, there exists an operator $N_n'\in\mathcal{L}\left[\mathfrak{H}\oplus\left(\bigoplus_{j=1}^{n}\mathfrak{H}_j\right)\right]$, similar to N, such that $\|N_n-N_n'\|$ is smaller than an arbitrarily small given $\varepsilon>0$. On the other hand, by Lemma 4,

$$\begin{bmatrix} N_n' & T_{n+1} \\ 0 & \lambda_{n+1}I_{n+1} \end{bmatrix}$$

can be approximated in the norm by operators similar to N_n'.

Since

$$\begin{bmatrix} N_n & T_{n+1} \\ 0 & \lambda_{n+1}I_{n+1} \end{bmatrix}-\begin{bmatrix} N_n' & T_{n+1} \\ 0 & \lambda_{n+1}I_{n+1} \end{bmatrix}=(N_n-N_n')\oplus 0_{n+1},$$

dist $[A, \mathcal{S}(N)]\leqq\|N_n-N_n'\|<\varepsilon$, whence the result follows. □

Proof of Theorem 1′. The last step of the proof is very similar to that of Lemma 4. Indeed, observe that if M is chosen as in Lemma 5 and W is an invertible operator in $\mathcal{L}(\mathfrak{H}')$, then

$$\begin{bmatrix} I & 0 \\ 0 & W \end{bmatrix}\begin{bmatrix} N & TW \\ 0 & M \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & W \end{bmatrix}^{-1}=\begin{bmatrix} N & T \\ 0 & WMW^{-1} \end{bmatrix}.$$

Since

$$\begin{bmatrix} N & TW \\ 0 & M \end{bmatrix}$$

can be uniformly approximated by operators similar to N (Lemma 5) and $\mathcal{S}(N)^-$ is invariant under similarities ([12]), we are done. □

4. The main result. The following result is our goal.

Theorem 2. *For every dimension $h\geqq\aleph_0$ there exists a universal quasinilpotent operator $Q_{uh}\in\mathcal{L}(\mathfrak{H})$*, dim $\mathfrak{H}=h$.

Proof. The proof combines the result of Theorem 1 with an argument due to N. Salinas ([5, Theorem 3.2]). Let $H_k\in\mathcal{L}(\mathfrak{H})$ be an hermitian operator such that $\Lambda(H_k)=\Lambda_h(H_k)=[0, 1/k]$ $(k=1, 2, \ldots)$. According to [9; 11], there exists a sequence $\{R_{kn}\}_{n=1}^{\infty}$ of nilpotent operators such that $\|H_k-R_{kn}\|<1/n$, $n=1, 2, \ldots$. By [16], there also exist nilpotent operators R_{kn}' similar to R_{kn}, such that $\|R_{kn}'\|<1/(k\cdot n)$.

Let Q_{uh} be an arbitrary quasinilpotent operator in $\mathcal{L}(\mathfrak{H})$, unitarily equivalent to $\bigoplus_{k,n=1}^{\infty} R_{kn}'$. Proceeding as in the proof of Lemma 2, we can see that $\mathcal{S}(Q_{uh})^-$ contains every operator unitarily equivalent to $R'_{kn}\oplus 0$ (for every fixed pair of values k and n). A fortiori, every $H'_k \approx H_k \oplus 0$ belongs to $\mathscr{S}(Q_{uh})^-$.

Let Q be an arbitrary quasinilpotent operator in $\mathscr{L}(\mathfrak{H})$. It follows from [2; 10; 11] that there exists an operator Q_k unitarily equivalent to

$$\begin{bmatrix} H'_k & T \\ 0 & L \end{bmatrix},$$

where $\Lambda(L) \subset \Lambda(H'_k) = \Lambda_h(H'_k) = [0, 1/k]$, such that $\|Q - Q_k\| < 2/k$. Since, by Theorem 1, $Q_k \in \mathscr{S}(H'_k)^- \subset \mathscr{S}(Q_{uh})^-$ for $k = 1, 2, \ldots$, it is easy to see that Q belongs to $\mathscr{S}(Q_{uh})^-$ too. □

5. Universal quasinilpotents for other closed bilateral ideals of $\mathscr{L}(\mathfrak{H})$. Let $\mathscr{J}_\alpha$ be a non-zero proper closed bilateral ideal of $\mathscr{L}(\mathfrak{H})$. Does there always exist a universal quasinilpotent for $\mathscr{J}_\alpha$? The answer is NO. Indeed, the existence of such universal operator depends on the cardinal α. Following [4; 6], we shall say that α is $\aleph_0$-regular if it cannot be written in the form $\alpha = \sum_{n=1}^{\infty} \alpha_n \; (= \sup_n \alpha_n)$ for a sequence $\{\alpha_n\}_{n=1}^{\infty}$ of cardinal numbers strictly smaller than α; α is called $\aleph_0$-irregular in the converse case. Now the complete answer to the above question is given by the following

Theorem 3. *Let* $\dim \mathfrak{H} = h \geqq \aleph_0$ *and let* $\mathscr{J}_\alpha$, $\aleph_0 \leqq \alpha \leqq h$, *be a proper closed bilateral ideal of* $\mathscr{L}(\mathfrak{H})$. *If neither*

(i) $\alpha = \aleph_{\nu+1}$ *for some ordinal* ν, *or*

(ii) α *is* $\aleph_0$*-irregular,*

then there exists a universal quasinilpotent operator $K_u = K_u(\alpha; h)$ *for* $\mathscr{J}_\alpha$.

On the other hand, if α *is an* $\aleph_0$*-regular limit cardinal, then* $\mathscr{S}(K)^- \subset \mathscr{J}_\beta$ *for some cardinal* β *strictly smaller than* α, *and therefore there is no universal quasinilpotent operator for* $\mathscr{J}_\alpha$.

Proof. Lemma 2 takes care of the case when $\alpha = \aleph_0$, so we can restrict our attention to the case $\alpha > \aleph_0$. We shall need the following auxiliary result.

Lemma 6. *Let* $\aleph_0 < \alpha \leqq h = \dim \mathfrak{H}$. *Then the closure of the set of all nilpotent operators in* $\mathscr{J}_\alpha$ *coincides with* $\mathscr{J}_\alpha \cap \mathscr{N}(\mathfrak{H})^-$. *In particular, this set contains every quasinilpotent element of* $\mathscr{J}_\alpha$.

Proof. Let $T \in \mathscr{J}_\alpha \cap \mathscr{N}(\mathfrak{H})^-$. Then there exist two sequences of operators, $\{T_n \colon \dim (T_n\mathfrak{H})^- = \alpha_n < \alpha\}_{n=1}^{\infty}$ and $\{Q_n \colon Q_n \in \mathscr{N}(\mathfrak{H})\}$ such that $\|T - T_n\| + \|T - Q_n\| < 1/n$. Proceeding as in [11] we can find a subspace $\mathfrak{H}_n$ of dimension $\alpha'_n = \max\{\alpha_n, \aleph_0\}$ reducing T_n and Q_n, such that $T_n|\mathfrak{H}_n^{\perp} = 0$. Clearly, $\|T_n|\mathfrak{H}_n - Q_n|\mathfrak{H}_n\| \leqq \|T_n - Q_n\| < 2/n$.

Let $R_n=(Q_n|\mathfrak{H}_n)\oplus(0|\mathfrak{H}_n^\perp)$. It readily follows that $R_n\in\mathcal{J}_\alpha$ and that $R_n^{k_n}=0$. if $Q_n^{k_n}=0$, i.e., R_n is a nilpotent element of $\mathcal{J}_\alpha$. Moreover, $\|T-R_n\|\leqq\|T-T_n\|+\|T_n-R_n\|<3/n$. Hence T is a norm limit of nilpotent elements of $\mathcal{J}_\alpha$. Therefore, $\mathcal{J}_\alpha\cap\mathcal{N}(\mathfrak{H})^-\subset\{Q\in\mathcal{J}_\alpha: Q \text{ is nilpotent}\}^-$. Since the converse inclusion is trivial, we have proved the first statement, the second one follows from [11]. □

Now we are in a position to finish the proof of Theorem 3. By Lemma 6, it will be enough to show that if $\alpha>\aleph_0$, then $\mathcal{S}(K_u)^-$ contains $\mathcal{J}_\alpha\cap\mathcal{N}(\mathfrak{H})^-$, for a suitable $K_u\in\mathcal{J}_\alpha$.

If α satisfies (i), $\mathcal{J}_\alpha=\{T\in\mathcal{L}(\mathfrak{H})\colon \dim(T\mathfrak{H})^-\leqq\aleph_\nu\}$ ([6; 14]) and the result follows as in Theorem 2; in fact, if $K\in\mathcal{J}_\alpha$ is nilpotent, then $\mathfrak{H}=\mathfrak{H}_0\oplus\mathfrak{H}_1$, where $\mathfrak{H}_0$, $\mathfrak{H}_1$ reduce K, $\dim\mathfrak{H}_0=\aleph_\nu$ and $K|\mathfrak{H}_1=0$. If $Q_u\in\mathcal{L}(\mathfrak{H}_0)$ is the operator defined in Theorem 2, then it readily follows that $K\in\mathcal{S}(Q_u\oplus 0)^-$, and $K_u=Q_u\oplus 0\in\mathcal{J}_\alpha$ is the solution to our problem. It α satisfies (ii), write $\mathfrak{H}=\bigoplus_{n=1}^{\infty}\mathfrak{H}_n$, where $\dim\mathfrak{H}_n=\alpha_n<\alpha$ and $\sum_{n=1}^{\infty}\alpha_n=\alpha$, and define $K_u=[\bigoplus_{n=1}^{\infty}(1/n)Q_{u\alpha_n}]$, where $Q_{u\alpha_n}$ is the universal quasinilpotent of Theorem 2 in dimension α_n. Clearly, K_u is a quasinilpotent element of $\mathcal{J}_\alpha$. Now the arguments of the proof of Theorem 2 and the results of [11] show that $\mathcal{S}(K_u)^-$ actually contains every nilpotent operator of $\mathcal{J}_\beta$ for every cardinal $\beta<\alpha$, and Lemma 3 and its proof show that $\mathcal{S}(K_u)^-$ also contains every nilpotent of $\mathcal{J}_\alpha$.

Let α be an $\aleph_0$-regular limit cardinal. Then, $\mathcal{J}_\alpha=\{T\in\mathcal{L}(\mathfrak{H})\colon \dim(T\mathfrak{H})^-<\alpha\}$ and, given $K\in\mathcal{J}_\alpha$, there exists a cardinal $\beta<\alpha$ such that $\dim(K\mathfrak{H})^-<\beta$ ([4]). Hence, $\mathcal{S}(K)^-\subset\mathcal{J}_\beta$, and this ideal is properly contained in $\mathcal{J}_\alpha$. Thus, if $T\in\mathcal{J}_\alpha\setminus\mathcal{J}_\beta$ and $A\in\mathcal{J}_\alpha$ is unitarily equivalent to

$$\begin{bmatrix}0 & T\\ 0 & 0\end{bmatrix},$$

then $A^2=0$, and A cannot belong to $\mathcal{S}(K)^-$. Therefore, there is no universal quasinilpotent operator for $\mathcal{J}_\alpha$. □

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