Acta Sci. Math., 38 (1976), 301—303

A remark on convergence systems in measure

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1. Preliminaries

Denote by $S=S(0, 1)$ the set of Lebesgue measurable almost everywhere finite functions on the interval (0, 1) with the complete metrizable topology of convergence in measure. In this paper "lim" will mean "limit in measure", unless stated otherwise explicitly.

Let $T=\|t_{i,j}\|$ be a matrix, not necessarily with rows of finite length, such that

(1) $$|t_{i,j}| \leqq K \quad (i, j = 1, 2, \dots), \quad \lim_{i\to\infty} t_{i,j} = 1 \quad (j = 1, 2, \dots).$$

Here and in the sequel K will denote some absolute constant not necessarily the same at each occurrence.

Finally, let B be a Banach space of sequences $a=\{a_1, a_2, \dots\}$ of real numbers such that for $a\in B$ we have

(2) $$|a_i| \leqq \|a\|_B, \quad a(N_1, N_2) = \{0, \dots, 0, a_{N_1}, a_{N_1+1}, \dots, a_{N_2}, 0, 0, \dots\}\in B, \quad \lim_{N_1\to\infty} \|a(N_1, N_2)\|_B = 0;$$

furthermore, if ε_j^i is a bounded double sequence of reals $(i, j=1, 2, \dots)$ such that $\lim_{i\to\infty} \varepsilon_j^i=1$ $(j=1, 2, \dots)$ then we have

(3) $$a^i = \{a_1\varepsilon_1^i, a_2\varepsilon_2^i, \dots\}\in B, \quad \lim_{i\to\infty} \|a^i - a\|_B = 0 \quad \text{for all} \quad a\in B.$$

For example l_p is such a space for $1\leqq p<\infty$.

The sequence $\{f_n\}\subset S$ is called a *T convergence system in measure for B* if the limit

(4) $$\hat{T}(a) = \lim_{i\to\infty} \lim_{N\to\infty} \tau_i^N(a) \quad \text{of} \quad \tau_i^N(a) = \sum_{j=1}^{N} t_{i,j}\cdot a_j f_j$$

Received October 16, 1975.

exists for all $a \in B$. In the special case, when $t_{i,j}=0$ for $i<j$ and $=1$ otherwise, $\{f_n\}$ is simply called a *convergence system in measure for B.*

Furthermore, the sequence $\{f_n\} \subset S$ is said to be *almost orthonormal* on the interval (0, 1) if for every $\varepsilon>0$ there exist a Lebesgue measurable set $E_\varepsilon \subset (0, 1)$, a constant M_ε depending only on ε, and an orthonormal system $\{\psi_n(\varepsilon, x)\}$ on (0, 1) such that mes $E_\varepsilon \geqq 1-\varepsilon$ and

$$f_n(x) = M_\varepsilon \psi_n(\varepsilon, x) \quad (x \in E_\varepsilon,\ n = 1, 2, \dots).$$

It is obvious that an almost orthonormal system is a convergence system in measure for l_2. In [2] NIKIŠIN proved the converse statement.

In [3] TANDORI proved the following generalization of Nikišin's result: If $\{f_n\} \subset S$ is a $(C, 1)$ convergence system in measure for l_2, that is if

$$\sum_{j=1}^{i} \left(1-\frac{j-1}{i}\right) a_j f_j = \sum_{j=1}^{i} t_{i,j} a_j f_j$$

converges in measure on (0, 1) as $n \to \infty$, for every $a \in l_2$, then $\{f_n\}$ is almost orthonormal.

Later on TANDORI [4] generalized this statement even to any summation method generated by a matrix $\|t_{i,j}\|$ having rows of finite lengths and satisfying conditions in (1).

In this paper we prove a theorem by which Tandori's general result follows from Nikišin's. Namely, in section 2 we are going to prove:

Theorem. *Under conditions* (1), (2), (3) *the system* $\{f_n\}$ $(\subset S)$ *is a T convergence system in measure for B if and only if it is a convergence system in measure for B.*

2.

We need the following Banach—Steinhaus type result.

Lemma. (See, e.g. [1] p. 52.) *Let E be a Banach space, F a metrizable topological vector space, and L_n continuous linear operators on E with values in F, converging at all points of E. Then the limit operator is also continuous and linear.*

In proving the Theorem first suppose $\{f_n\} \subset S$ is a T convergence system in measure for B. Apply the Lemma twice, first for fixed i to the sequence $\{\tau_i^N\}$ of operators in (4), which are continuous on account (3). Denoting by τ_i the limit operators and applying the Lemma to this sequence we obtain that the linear operator $\hat{T}$ in (4) is continuous. Let $a \in B$ be arbitrary. We have to prove the existence of the limit

$$\lim_{N \to \infty} \sum_{j=1}^{N} a_j f_j .$$

According to the completeness of S it is enough to prove that

$$\lim_{N_1, N_2\to\infty} \sum_{j=N_1}^{N_2} a_j f_j = 0.$$

But this follows from the continuity of $\hat{T}$ at the zero element of B, using

$$\hat{T}(a(N_1, N_2)) = \sum_{j=N_1}^{N_2} a_j f_j.$$

Conversely, suppose $\{f\}$ is a convergence system in measure for B. Then we obtain similarly that the linear operator

$$L(a) = \lim_{N\to\infty} \sum_{j=1}^{N} a_j f_j$$

is continuous. Using (3) for $\varepsilon_j^i = t_{i,j}$, this shows that

$$\lim_{i\to\infty} L(a^i) = L(\lim_{i\to\infty} a^i) = L(a).$$

This completes the proof.

References

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