On the strong approximation of Fourier series

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1. Let $f(x)$ be a continuous and 2π-periodic function and let

$$(1.1) \qquad f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be its Fourier series. Let $s_n(x) = s_n(f; x)$ and $\sigma_n^\alpha(x) = \sigma_n^\alpha(f; x)$ denote the n-th partial sum and the (C, α)-mean of (1.1), and let $\tilde{f}(x)$, $\tilde{s}_n(x)$, $\tilde{\sigma}_n^\alpha(x)$ denote the conjugate functions, respectively.

In [2] we investigated among others the means

$$V_n(f, \lambda, p; x) = \left\{ \frac{1}{\lambda_n} \sum_{k=n-\lambda_n}^{n-1} |s_k(x) - f(x)|^p \right\}^{1/p},$$

where $\lambda = \{\lambda_n\}$ is a nondecreasing sequence of integers such that $\lambda_1 = 1$ and $\lambda_{n+1} - \lambda_n \leq 1$, and $p > 0$. Such a mean is called a "generalized strong de la Vallée Poussion mean", or briefly, a *strong (V, λ)-mean*.

In [2] we proved the following theorems:

Theorem A. *If $n = O(\lambda_n)$ and $p > 0$, then*

$$(1.2) \qquad V_n(f, \lambda, p; x) = O(E_{n-\lambda_n})$$

polds uniformly, where $E_n = E_n(f)$ denotes the best approximation of f by trigonometric holynomials of order at most n.

Theorem B. *Suppose that $f(x)$ r times derivable and $f^{(r)} \in Lip\ \alpha$ $(0 < \alpha \leq 1)$, and that $n = O(\lambda_n)$. Then for any $p > 0$*

$$(1.3) \quad V_n(f, \lambda, p; x) = \begin{cases} O\left(\dfrac{1}{n^{r+\alpha}}\right) & \text{for} \quad (r+\alpha)p < 1, \\[4mm] O\left(\dfrac{1}{n^{r+\alpha}}\left(1 + \log \dfrac{n}{n-\lambda_n+1}\right)^{1/p}\right) & \text{for} \quad (r+\alpha)p = 1, \end{cases}$$

uniformly. The same estimate holds for $V_n(\tilde{f}, \lambda, p; x)$. *Furthermore, if* $(r+\alpha)p = 1$ $(0 < \alpha < \leqq 1)$, *then there exist functions* $f_1(x)$ *and* $f_2(x)$ *such that their r-th derivatives exist and belong to* Lip α, *moreover, both*

$$\varlimsup_{n \to \infty} V_n(f_1, \lambda, p; 0) \quad and \quad \varlimsup_{n \to \infty} V_n(\tilde{f}_2; \lambda, p; 0) \quad are \quad \geqq \frac{c}{n^{r+\alpha}} \left(1 + \log \frac{n}{n - \lambda_n + 1}\right)^{1/p},$$

where $c(>0)$ *is independent of n.*

In this paper we generalize these results. Among others we omit the restriction $n = O(\lambda_n)$, but then the estimations will not be necessarily best possible, and show that there exists a function f_0 such that both $f_0^{(r)}$ and $\tilde{f}_0^{(r)}$ belong to the class Lip 1 and the estimations (1.3) are best possible for the means $V_n(f_0, \lambda, p; x)$ also. Furthermore we show that if $0 < \alpha < 1$ then the partial sums in the means $V_n(f, \lambda, p; x)$ can be replaced by (C, β)-means of negative order.

More precisely we prove the following theorems:

Theorem 1. *For any positive p we have*

$$(1.4) \qquad\qquad V_n(f, \lambda, p; x) = O\left(\left(\frac{n}{\lambda_n}\right)^{1/p} E_{n-\lambda_n}\right)$$

uniformly.

Theorem 2. *If* $f^{(r)} \in$ Lip α $(0 < \alpha \leqq 1)$, *then for any* $p > 0$

$$(1.5) \quad V_n(f, \lambda, p; x) = \begin{cases} O\left(\left(\dfrac{n}{\lambda_n}\right)^{1/p} \dfrac{1}{n^{r+\alpha}}\right) & for \quad (r+\alpha)p < 1, \\[3mm] O\left(\dfrac{1}{\lambda_n^{r+\alpha}} \left(1 + \log \dfrac{n}{n - \lambda_n + 1}\right)^{1/p}\right) & for \quad (r+\alpha)p = 1, \\[3mm] O(\lambda_n^{-1/p}(n - \lambda_n + 1)^{\frac{1}{p} - r - \alpha}) & for \quad (r+\alpha)p > 1, \end{cases}$$

holds uniformly. The same estimate also holds for $V_n(\tilde{f}, \lambda, p; x)$.

Theorem 3. *Suppose that* $0 < \alpha \leqq 1$, $p > 0$, *and* $n = O(\lambda_n)$. *Then there exists* f_0 *such that* $f_0^{(r)}$ *and* $\tilde{f}_0^{(r)}$ *belong to the class* Lip α, *and still*

$$(1.6) \qquad \varlimsup_{n \to \infty} V_n(f_0, \lambda, p; 0) \geqq \begin{cases} dn^{-r-\alpha} & if \quad (r+\alpha)p < 1, \\[2mm] dn^{-r-\alpha}\left(1 + \log \dfrac{n}{n - \lambda_n + 1}\right)^{1/p} & if \quad (r+\alpha)p = 1, \\[2mm] dn^{-1/p}(n - \lambda_n + 1)^{1/p - r - \alpha} & if \quad (r+\alpha)p > 1, \end{cases}$$

where $d = d(\lambda, p) > 0$.

Theorem 4. *Suppose that $f \in \text{Lip } \alpha$ for some $0 < \alpha < 1$, that $\beta > -1/2$ and that the positive number p satisfies the inequality $p\beta > -1$. Then we have, uniformly,*

$$(1.7) \qquad \left[\frac{1}{\lambda_n} \sum_{k=n-\lambda_n}^{n} |\sigma_k^\beta(x) - f(x)|^p \right]^{1/p} = \begin{cases} O\left(\left(\frac{n}{\lambda_n} \right)^{1/p} \frac{1}{n^\alpha} \right) \\ O\left(\frac{1}{\lambda_n^\alpha} \left(1 + \log \frac{n}{n - \lambda_n + 1} \right) \right)^{1/p} \\ O\left(\lambda_n^{-1/p} (n - \lambda_n + 1)^{1/p - \alpha} \right) \end{cases}$$

according as αp is < 1, $= 1$, or > 1.

In what follows $\| \cdot \|$ and $[\cdot]$ denote supremum norm and integral part, respectively, and $\omega(f; \delta)$ denotes the modulus of continuity of f.

Finally we improve one part of the following theorem of SZABADOS [7]:

Theorem C. *If $0 < p < 1$ and $r = [1/p]$, then the condition*

$$(1.8) \qquad \left\| \sum_{n=0}^{\infty} |s_n(x) - f(x)|^p \right\| \leq K$$

implies that $f^{(r-1)}(x)$ is continuous and

$$\omega(f^{(r-1)}; h) = \begin{cases} O\left(h \left(\log \frac{1}{h} \right)^{1/p} \right) & \text{if } \frac{1}{p} = r, \\ O(h) & \text{otherwise.} \end{cases}$$

We have the following

Theorem 5. *If $0 < p < 1$ and $1/p - r = \alpha > 0$, then condition (1.8) implies that $f^{(r)}$ is continuous and*

$$(1.9) \qquad \omega(f^{(r)}, h) = O\left(h^\alpha \left(\log \frac{1}{h} \right)^{1/p - 1} \right).$$

In connection with these results we formulate the following

Conjecture. *).* *If $0 < p < 1$ and $1/p = r + \alpha$, then condition (1.8) implies that*

$$(1.10) \qquad \omega(f^{(r-1)}; h) = O\left(h \log \frac{1}{h} \right) \quad \text{if } \alpha = 0,$$

and

$$(1.11) \qquad \omega(f^{(r)}; h) = O(h^\alpha) \quad \text{if } \alpha > 0.$$

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*) Added in proof: This conjecture has been verified by the author.

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Finally we remark that the estimations (1.10) and (1.11) are, in general, best possible. Namely, if $1/p = r + \alpha$ and r is an odd integer, then the function

$$f_0(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^{1+1/p}}$$

has $(r-1)$-th and r-th derivatives such that if $\alpha = 0$ then

$$\left| f_0^{(r-1)}\left(\frac{\pi}{2^n}\right) - f_0^{(r-1)}(0) \right| > \frac{1}{8} \frac{\pi}{2^n} \log \frac{2^n}{\pi} \quad \text{for all} \quad n \geq 6,$$

(see [5], pp. 224—227); and since

$$f_0^{(r)}(x) = \pm \sum_{n=1}^{\infty} \frac{\cos nx}{n^{1+\alpha}} \quad (\alpha > 0),$$

the inequality $\omega(f_0^{(r)}, h) \geq c\, h^{\alpha} (c > 0)$ is obvious. Furthermore a standard computation (see e.g. [5], pp. 225—226) shows that for this function f_0 (1.8) holds.

2. To prove our theorems we require three lemmas.

Lemma 1. ([2], Lemma 2) *If* $g \in L(0, 2\pi)$ *and* $|g(x)| \leq M$ *for all* x, *then, for any* $q > 0$, *we have*

$$\frac{1}{m} \sum_{k=1}^{m} |s_k(g; x)|^q \leq C_q^q M^q.$$

Lemma 2. ([3], Lemma) *If* $f \in \mathrm{Lip}\, \gamma$, $0 < \gamma < 1$, $\delta > -1/2$, *and if the positive number* p *satisfies the inequality* $p\delta > -1$, *then we have for any* $n(\geq 1)$

$$\frac{1}{n} \sum_{v=n}^{2n} |\sigma_v^{\delta}(f; x) - \sigma_v^{\delta+1}(f; x)|^p = O(n^{-\gamma p}).$$

Lemma 3. ([2], estimate (6), p. 150)*We have for any* $q > 0$ *and* n

$$h_n(f, q; x) \equiv \left(\frac{1}{n} \sum_{v=n}^{2n} |s_v(f, x) - f(x)|^q \right)^{1/q} = O(E_n).$$

3. Proof of Theorem 1. Let T_m^* denote the trigonometric polynomial of best approximation to f of order at most m. From the definition of s_n it is clear that if $v \geq m$ then $s_v(f - T_m^*; x) = s_v(f; x) - T_m^*(x)$. Using this we have

$$\left(\frac{1}{\lambda_n} \sum_{v=n-\lambda_n}^{n-1} |s_v(x) - f(x)|^p \right)^{1/p} \leq \left[\frac{2^p}{\lambda_n} \sum_{v=n-\lambda_n}^{n-1} \left(|s_v(f - T_{n-\lambda_n}^*; x)|^p + |T_{n-\lambda_n}^*(x) - f(x)|^p \right) \right]^{1/p} \leq$$

(3.1)

$$\leq 2^{1+1/p} \left(\left\{ \frac{n}{\lambda_n} \cdot \frac{1}{n} \sum_{v=n-\lambda_n}^{n-1} |s_v(f - T_{n-\lambda_n}^*; x)|^p \right\}^{1/p} + E_{n-\lambda_n} \right).$$

Applying Lemma 1 (with $g = f - T_{n-\lambda_n}^*$ and $q = p$) we immediately obtain the statement of Theorem 1.

Proof of Theorem 2. By the well-known theorem of Jackson the assumption $f^{(r)} \in \text{Lip } \alpha \ (0 < \alpha \leq 1)$ implies that

$$E_n(f) = O(n^{-r-\alpha}) \quad \text{and} \quad E_n(\tilde{f}) = O(n^{-r-\alpha}).$$

Hence, by Lemma 3, we obtain that

(3.2) $$h_n(f, p; x) = O(n^{-r-\alpha}) \quad \text{and} \quad h_n(\tilde{f}, p; x) = O(n^{-r-\alpha}).$$

If $2^{m_1} \leq n - \lambda_n < 2^{m_1+1}$ and $2^{m_2} < n \leq 2^{m_2+1}$ then, by (3.2), we have

$$\frac{1}{\lambda_n} \sum_{v=n-\lambda_n}^{n-1} |s_v(x) - f(x)|^p \leq \frac{1}{\lambda_n} \sum_{m=m_1}^{m_2} \sum_{v=2^m}^{2^{m+1}-1} |s_v(x) - f(x)|^p \leq$$

$$\leq \frac{O(1)}{\lambda_n} \sum_{m=m_1}^{m_2} 2^{m(1-p(r+\alpha))} \equiv \Sigma_1.$$

Now,

$$\Sigma_1 \leq O(1) \frac{1}{\lambda_n} 2^{m_2(1-p(r+\alpha))} = O\left(\frac{n}{\lambda_n} \cdot \frac{1}{n^{p(r+\alpha)}}\right), \quad \text{if } p(r+\alpha) < 1,$$

$$\Sigma_1 \leq O(1) \frac{1}{\lambda_n} (m_2 - m_1) = O\left(\frac{1}{\lambda_n}\left(1+\log\frac{n}{n-\lambda_n+1}\right)\right), \quad \text{if } p(r+\alpha) = 1,$$

$$\Sigma_1 = O(\lambda_n^{-1}(n-\lambda_n+1)^{1-p(r+\alpha)}), \quad \text{if } p(r+\alpha) > 1.$$

Whence (1.5) obviously follows.

The proof for $\tilde{f}$ runs similarly.

Proof of Theorem 3. Set

$$f_0(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{n\alpha}} \sum_{l=2^{n-1}+1}^{2^n} \left(\frac{\cos(5\cdot 2^n - l)x}{(5\cdot 2^n - l)l} - \frac{\cos(5\cdot 2^n + l)x}{(5\cdot 2^n + l)l}\right).$$

In [4] (Theorem 1) it is proved that $f_0^{(r)}$ and $\tilde{f}_0^{(r)}$ belong to the class Lip α if $\alpha=1$, furthermore in [1] this statement in the case $\alpha < 1$ with an odd r is verified. Thus we only have to show that $f_0^{(r)} \in \text{Lip } \alpha$ if r is an even integer and $0 < \alpha < 1$. In this case

$$f_0^{(r)}(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+\frac{r}{2}+1}}{2^{n\alpha}} \sum_{l=2^{n-1}+1}^{2^n} \left(\frac{\cos(5\cdot 2^n - l)x}{l} - \frac{\cos(5\cdot 2^n + l)x}{l}\right) \equiv$$

$$\equiv \sum_{n=1}^{\infty} \frac{(-1)^{n+\frac{r}{2}+1}}{2^{n\alpha}} R_n(x),$$

where $\|R_n(x)\| \leq 2$. Thus, if $4\cdot 2^m \leq n < 4\cdot 2^{m+1}$, then

$$E_n(f_0^{(r)}) \leq \|f_0^{(r)}(x) - s_n(f_0^{(r)}; x)\| \leq 2 \sum_{k=m}^{\infty} \frac{1}{2^{k\alpha}} = O\left(\frac{1}{n^\alpha}\right),$$

which implies $f_0^{(r)} \in \text{Lip } \alpha \ (0 < \alpha < 1)$.

In the proof of (1.6) we distinguish two cases according as the sequence $\left\{\dfrac{n}{n-\lambda_n}\right\}$ is bounded or not. First we investigate the bounded case. Let $n=12\cdot 2^m$ and let $m_1=\max\,(n-\lambda_n,\,22\cdot 2^{m-1})$, $m_2=\max\,(m_1,\,23\cdot 2^{m-1})$ and $m_3=\max\left(m_2,\,n-\left[\dfrac{\lambda_n+1}{2}\right]\right)$.
Then

$$V_n(f_0,\,\lambda,\,p;\,0)=\left\{\frac{1}{\lambda_n}\sum_{\nu=n-\lambda_n}^{n-1}|s_\nu(0)-f_0(0)|^p\right\}^{1/p}\geqq$$

$$\geqq\left\{\frac{1}{\lambda_n}\left(\sum_{\nu=m_1}^{m_2-1}+\sum_{\nu=m_2}^{m_3}\right)\left|\frac{1}{n^\alpha}\sum_{l=\nu-10\cdot 2^m+1}^{2^m+1}\frac{1}{n^r l}\right|^p\right\}^{1/p}.$$

Hence, by $n=0(\lambda_n)$, it follows that

$$\sum_{\nu=m_1}^{m_2-1}\left|\frac{1}{n^\alpha}\sum_{l=\nu-10\cdot 2^m+1}^{2^m+1}\frac{1}{n^r l}\right|^p\geqq(m_2-m_1)\left|\frac{1}{n^\alpha}\sum_{l=m_2-10\cdot 2^m+1}^{2^m+1}\frac{1}{n^r l}\right|^p\geqq$$

$$\geqq(m_2-m_1)\left|\frac{1}{n^{\alpha+r+1}}(n-m_2)\right|^p\geqq d_1(p,\,\lambda)(m_2-m_1)\frac{1}{n^{(\alpha+r)p}},$$

and

$$\sum_{\nu=m_2}^{m_3}\left|\frac{1}{n^\alpha}\sum_{l=\nu-10\cdot 2^m+1}^{2^m+1}\frac{1}{n^r l}\right|^p\geqq(m_3-m_2)\left|\frac{1}{n^{\alpha+r+1}}(n-m_3)\right|^p\geqq d_2(p,\,\lambda)(m_3-m_2)\frac{1}{n^{(\alpha+r)p}}.$$

Thus we obtain that

$$V_n(f_0,\,\lambda,\,p;\,0)\geqq d_3(p,\,\lambda)\left[(m_3-m_1)\frac{1}{\lambda_n}\cdot\frac{1}{n^{(\alpha+r)p}}\right]^{1/p}\geqq d_4(p,\,\lambda)\frac{1}{n^{r+\alpha}},$$

which proves the statements of (1.6) under the assumption that the sequence $\left\{\dfrac{n}{n-\lambda_n}\right\}$ is bounded.

If $\left\{\dfrac{n}{n-\lambda_n+1}\right\}$ is not bounded, then we may suppose that there exist infinitely many n with $4\cdot 2^m<n\leqq 4\cdot 2^{m+1}$ and $4\cdot 2^\mu\leqq n-\lambda_n+4<4\cdot 2^{\mu+1}$ such that $m>\mu+2$. Then

(3.3)
$$V_n(f_0,\,\lambda,\,p;\,0)^p\geqq\frac{1}{\lambda_n}\sum_{i=\mu+1}^{m-1}\sum_{\nu=4\cdot 2^i+1}^{4\cdot 2^{i+1}}|s_\nu(0)-f_0(0)|^p\geqq$$

$$\geqq\frac{1}{\lambda_n}\sum_{i=\mu+1}^{m-1}\sum_{\nu=11\cdot 2^{i-1}+1}^{12\cdot 2^{i-1}}|s_\nu(0)-f_0(0)|^p\equiv\frac{1}{\lambda_n}\sum_{i=\mu+1}^{m-1}I_i.$$

I_i can be estimated as follows

$$I_i \geqq \sum_{v=11\cdot2^{i-1}}^{23\cdot2^{i-2}} \left(\frac{1}{2^{i\alpha}} \sum_{l=v-10\cdot2^{i-1}+1}^{2^i} \frac{1}{6^r 2^{ir}l} \right)^p \geqq \sum_{v=11\cdot2^{i-1}}^{23\cdot2^{i-2}} \left(\frac{1}{2^{i\alpha}} \sum_{l=3\cdot2^{i-2}+1}^{2^i} \frac{1}{6^r 2^{ir}l} \right)^p \geqq$$

$$\geqq d_1(p, r) 2^{i-2} \frac{1}{2^{i(r+\alpha)p}} = d_2(p, r) 2^{i(1-(r+\alpha)p)}.$$

Hence and from (3.3) we obtain that

$$V_n(f_0, \lambda, p; 0) \geqq d_3(p, r) \left(\frac{1}{\lambda_n} \sum_{i=\mu+1}^{m-1} 2^{i(1-(r+\alpha)p)} \right)^{1/p},$$

whence (1.6) can be deduced by an easy calculation.

The proof of Theorem 3 is thus completed.

Proof of Theorem 4. It is clear that

$$(3.4) \quad \frac{1}{\lambda_n} \sum_{k=n-\lambda_n}^{n} |\sigma_k^\beta(x) - f(x)|^p \leqq \frac{K}{\lambda_n} \sum_{k=n-\lambda_n}^{n} (|\sigma_k^\beta(x) - \sigma_k^{\beta+1}(x)|^p + |\sigma_k^{\beta+1}(x) - f(x)|^p) \equiv$$

$$\equiv \Sigma_1 + \Sigma_2.$$

It is known (see e.g. [1] Theorem 3) that $f(x) \in \text{Lip } \alpha$ implies

$$|\sigma_k^{\beta+1}(x) - f(x)| = O(k^{-\alpha}) \quad (\beta > -\tfrac{1}{2}),$$

whence

$$(3.5) \qquad \Sigma_2 = O \left(\frac{1}{\lambda_n} \sum_{k=n-\lambda_n}^{n} k^{-\alpha p} \right).$$

Furthermore,

$$(3.6) \qquad \Sigma_1 = \frac{1}{\lambda_n} \left(\sum_{k=n-\lambda_n}^{n/2} + \sum_{k=n/2}^{n} \right) |\sigma_k^\beta(x) - \sigma_k^{\beta+1}(x)|^p = \Sigma_3 + \Sigma_4. \ ^1)$$

By Lemma 2

$$(3.7) \qquad \Sigma_4 = O \left(\frac{1}{\lambda_n} n^{1-\alpha p} \right)$$

and if $2^\mu \leqq n - \lambda_n < 2^{\mu+1}$ and $2^{\mu_1} < n/2 \leqq 2^{\mu_1+1}$, then

$$(3.8) \qquad \Sigma_3 \leqq \frac{1}{\lambda_n} \sum_{m=\mu}^{\mu_1} \sum_{k=2^m}^{2^{m+1}} |\sigma_k^\beta(x) - \sigma_k^{\beta+1}(x)|^p \leqq \frac{1}{\lambda_n} \sum_{m=\mu}^{\mu_1} 2^{m(1-\alpha p)}.$$

Collecting the estimates (3.4), (3.5), (3.6), (3.7) and (3.8) an easy calculation gives the statements of (1.7), which is the required proof.

$^1)$ $\sum_{n=a}^{b}$, where a and b are not integers, means a sum over all integers between a and b; if $b < a$ then the sum means zero.

Proof of Theorem 5. The proof runs on analogous lines as that of Szabados. Using the Lebesgue's estimate and (1.8) we obtain

$$E_{2n} \leqq \left\| \frac{1}{n+1} \sum_{k=n}^{2n} s_k(x) - f(x) \right\| \leqq$$

$$\leqq \frac{1}{n} \left\| \sum_{k=n}^{2n} |s_k(x) - f(x)|^p |s_k(x) - f(x)|^{1-p} \right\| \leqq K_1 \frac{1}{n} (E_n \log n)^{1-p},$$

whence, by a standard computation (see inequality (8) in [7]),

(3.9) $E_n^p = O\big(n^{-1}(\log n)^{1-p}\big)$

follows. Using the estimate ([6], Theorem 8, p. 61)

$$E_n(f^{(r)}) \leqq K(r) \sum_{k=[n/2]}^{\infty} k^{r-1} E_k(f),$$

(3.9) implies that

$$E_n(f^{(r)}) = O\left(\frac{(\log n)^{1/p-1}}{n^\alpha} \right),$$

whence, according to the inequality ([6], Theorem 4, p. 59)

$$\omega(f, h) \leqq Kh \sum_{n=0}^{1/h} E_n(f)$$

we get

$$\omega(f^{(r)}, h) \leqq Kh \sum_{n=1}^{1/h} \frac{(\log n)^{1/p-1}}{n^\alpha} \leqq K_1 h^\alpha \left(\log \frac{1}{h} \right)^{1/p-1}$$

which completes the proof.

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