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Spectral mapping theorems for semigroups of operators

B. NAGY

1. Introduction and notations

Spectral mapping theorems for essential spectra have been investigated by B. GRAMSCH and D. LAY [1] even in the case if T is a closed unbounded linear operator with nonvoid resolvent set. However, their results do not apply to the essential spectra of semigroups of linear operators, for in this case the mapping f is not locally holomorphic on a neighborhood of the extended spectrum of T. The aim of this paper is to extend the results of [1] to semigroups of linear operators in Banach spaces.

Let X, Y denote complex Banach spaces, $B(X, Y)$ the space of bounded linear operators from X to Y and set $B(X)=B(X, X)$. We shall always assume that the semigroup $\{T(t),\ t>0\}\subset B(X)$ is of class (A), and additional restrictions will be explicitly stated (cf. [2, pp. 321—323]). A will denote the infinitesimal generator of $T(t)$.

Let V be a closed linear operator with domain $D(V)\subset X$ and range $R(V)\subset X$. Suppose that the resolvent set $\varrho(V)$ of V is nonvoid. The nullity of V, $n(V)$ is the dimension of the kernel $N(V)$. The defect of V, $d(V)$ is the algebraic dimension of the quotient vector space $X/R(V)$. The index of V, ind (V) is $n(V)-d(V)$, where $\infty-\infty$ is undefined. The ascent of V, $a(V)$ is the smallest nonnegative integer p such that $N(V^p)=N(V^{p+1})$. The descent of V, $e(V)$ is the smallest nonnegative integer q with $R(V^q)=R(V^{q+1})$. If no such p or q exist, set $a(V)=\infty$ or $e(V)=\infty$, respectively.

A comprehensive survey of the essential spectra of V has been given in [1]. To unify notation, we shall define them by means of regularity sets G_i ($i=1, 2, \dots, 11$) as follows. $V\in G_i$ if and only if

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G_1 : $V^{-1} \in B(X)$,
G_2 : ind $(V)=0$ and $a(V)=e(V)<\infty$,
G_3 : ind $(V)=0$,
G_4 : ind (V) is finite,
G_5 : $n(V)<\infty$ and $R(V)=R(P)$ for some $P=P^2 \in B(X)$,
G_6 : $d(V)<\infty$ and $N(V)=R(P)$ for some $P=P^2 \in B(X)$,
G_7 : $n(V)<\infty$ and $R(V)$ is closed,
G_8 : $d(V)<\infty$,
G_9 : $G_7 \cup G_8$,
G_{10}: $R(V)$ is closed,
G_{11}: $a(V)<\infty$ and $e(V)<\infty$.

We shall omit nomenclature, for it is not unified in the literature. It is clear that the following relations hold:

$$\begin{array}{c} G_1 \subset G_2 \subset G_3 \subset G_4 \subset \left\{ \begin{matrix} G_6 \subset G_8 \\ G_5 \subset G_7 \end{matrix} \right\} \subset G_9 \subset G_{10}. \\ \hspace{-17em}\cap \\ \hspace{-17em}G_{11} \end{array}$$

We remark that the following example shows that in general we do not have $B(X) \cap G_{11} \subset G_{10}$.

Example. Put $X=l_2$ and for $x=(x_1, x_2, \dots) \in X$ define $Vx=(0, x_1, 0, 1/3x_3, 0, 1/5x_5, 0, \dots)$. Then V is compact and $\dim R(V)=\infty$, hence $R(V)$ is not closed. Further $V^2=0$, thus $a(V)=2$, and $e(V)=2$, contrary to the required containment relation.

The essential spectrum $s_i(V)$ is the set of complex numbers c such that $V-c=V-cI \notin G_i$ ($i=2, 3, \dots, 11$; for $i=1$ we get the spectrum of V). We emphasize that $s_i(V)$ is a subset of the proper complex plane ($i=1, 2, \dots, 11$), contrary to the definitions of [1, pp. 30—31]. In what follows we intend to prove mapping theorems of the type

$$\exp[ts_i(A)] \subset s_i[T(t)],$$

which is well-known for $i=1$. Theorem 10 will show that the converse relations as a rule cannot be expected to be true.

We remark that a projection operator will always be understood to belong to $B(X)$, X^* will denote the adjoint space of X and V^* the adjoint of the operator V.

From the method of the proofs it will be seen that, according to the results of [9, pp. 285—286], some spectral mapping theorems for essential spectra of cosine operator functions can be proved by a similar method. In this connection we take the opportunity to note that Theorem 3 in [9, p. 285] has been misstated, and the following should be substituted for it. The author apologizes for the error.

Theorem 3. [9] *If C is a cosine operator function, A its generator and $s\in R$, then* $\operatorname{ch}\{s\sqrt{\sigma(A)}\}\subset\sigma\{C(s)\}$. *Further, if a is complex number and a^2-A has the spectral property P_v $(v=1, 2, 3)$, then so does* $\operatorname{ch}(as)-C(s)$.

The proof remains unchanged.

2. Spectral mapping theorems

Let $T(t)$ be a semigroup of class (A), and A its infinitesimal generator. In what follows we will heavily rely on the definitions and results of the operational calculus and spectral theory as developed in [2, Chapters 15, 16]. The most relevant results are summarized in the following lemma (cf. [2, Theorem 16.6.1]).

Lemma 1. *Let φ be a real-valued Borel measurable submultiplicative function on $[0, \infty)$ with $\varphi(0)=1$. Suppose $g\in\mathscr{S}(\varphi)$, $A\prec\varphi$, then the linear operator $F(g; A)$ defined by*

$$F(g;\, A)x = \int_0^\infty T(t)x\,dg(t)$$

for $x\in X_1(A)=\{x\in X;\ \lim_{t\to 0+} T(t)x=x\}$ has a unique bounded linear extension $F(g)\in \in B(X)$. Moreover, the complex function

$$f(g;\, c) = \int_0^\infty e^{ct}dg(t)$$

is defined and holomorphic for $\operatorname{Re} c<w_0=\lim_{t\to\infty} t^{-1}\log\varphi(t)$. *Suppose $a\in\mathscr{S}(\varphi)$, $A\prec\varphi$, $c\in s_1(A)$. Then there exist a submultiplicative φ' such that $A\prec\varphi'$, $\mathscr{S}(\varphi)\subset\mathscr{S}(\varphi')$, and an element $b\in\mathscr{S}(\varphi')$ such that*

$$F(a)-f(a;\, c) = (A-c)\,F(b), \tag{1}$$

$$[F(a)-f(a;\, c)]x = F(b)(A-c)x \quad \text{for} \quad x\in D(A). \tag{2}$$

The most important special case is described in

Lemma 2. *For every $t>0$ and c complex,*

$$T(t)-e^{ct} = (A-c)\,F, \tag{3}$$

$$[T(t)-e^{ct}]x = F(A-c)x \quad \text{for} \quad x\in D(A), \tag{4}$$

where $F\in B(X)$, and for $x\in X_1(A)$

$$Fx = F_c^t x = e^{ct}\int_0^t e^{-cs}T(s)x\,ds. \tag{5}$$

Further, if $T(t)$ is of class $(1, A)$, then (5) holds for every $x\in X$.

Proof. A closer inspection of the proof of [2, Theorem 16.6.1] shows that if $a=e_t$, i.e. $F(a)=T(t)$, $f(a;c)=e^{ct}$, then (1) and (2) hold for every complex c, and $F=F(b)$ is given by (5) for $x\in X_1(A)$. Moreover, if $T(t)$ is of class $(1, A)$, then the right side of (5) is defined for every $x\in X$, and

$$\left\|\int_0^t e^{-cs}T(s)x\,ds\right\| \leqq \int_0^t |e^{-cs}|\,\|T(s)\|\,ds\,\|x\| \leqq K\|x\|.$$

Thus the operators on both sides of (5) are bounded and coincide on the dense set $X_1(A)$, hence on all of X.

Remark. In what follows we will prove theorems of the following type: $c\in s_i(A)$ implies $e^{ct}\in s_i(T(t))$ for $t>0$, or equivalently,

$$(6)\qquad T(t)-e^{ct}\in G_i \quad\text{implies}\quad A-c\in G_i \quad (t>0).$$

Since for every complex number c the operator $B=A-c$ is the infinitesimal generator of the semigroup $S(t)=e^{-ct}T(t)$ of the same class (see [2, pp. 357—359]), we may and will restrict ourselves in the statements and proofs to the case $c=0$ in (6). For a fixed $t>0$ we shall often write, for the sake of brevity,

$$(7)\qquad V=T(t)-I,\quad V_0=FA.$$

Theorem 1. *If $T(t)-I\in G_7$, then $A\in G_7$.*

Proof. To avoid trivialities we assume $\dim X=\infty$. Since $V\supset FA$, therefore $N(A)\subset N(V)$, hence $n(A)<\infty$. By assumption, there is a projection P of X onto $N(V)$, i.e.

$$X=PX\oplus(I-P)X=N(V)\oplus X',$$

where $P\neq I$. $R(V)$ is closed, thus for $x\in X$

$$\|Vx\|\geqq q\cdot\operatorname{dist}(x, N(V))=q\|x-n\|,$$

where $q>0$ and $n\in N(V)$. Hence

$$\|Vx\|\geqq\frac{q}{\|I-P\|}\|(I-P)x\|=q'\|(I-P)x\|.$$

For $x\in D(A)$ we get $\|F\|\cdot\|Ax\|\geqq\|Vx\|\geqq q'\|(I-P)x\|$. The equality $F=0$ would imply $N(V)=X$, a contradiction, thus for $x\in D(A)\cap X'$ we have

$$(8)\qquad \|Ax\|\geqq r\|x\|\geqq r\cdot\operatorname{dist}(x, N(A|X'))\quad (r>0),$$

where $A|X'$ denotes the restriction of A to $D(A)\cap X'$. The set $X'=(I-P)X$ is a closed subspace of X, hence $A|X'$ is a closed operator. By (8), $A|X'$ has closed range, and again [3, Lemma 333] yields that $R(A)$ is closed, hence $A\in G_7$.

Theorem 2. *$T(t)-I\in G_8$ implies $A\in G_8$.*

Proof. Since $R(V)=R(AF)\subset R(A)$, we obtain codim $R(A)\leqq$ codim $R(V)$, i.e. $d(A)\leqq d(V)$, and the assertion follows immediately.

From these results we obtain the following

Corollary 1. *$T(t)-I\in G_i$ implies $A\in G_i$ $(i=4, 9)$.*

Theorem 3. *If $T(t)$ is of class $(1, A)$ and $T(t)-I\in G_{11}$, then $A\in G_{11}$.*

Proof. Since $D(V)=X$, the assumption implies that $a(V)=e(V)=p$, by [5, Theorem 5.41—E]. If $p=0$, then $1\in\varrho(T(t))$, hence $A\in G_1\subset G_{11}$, by [2, Theorem 16.7.1]. If $p>0$, then [4, Theorem 2.1] yields that 1 is a pole of the resolvent operator $R(c; T(t))$ of order p. Then there exists a deleted neighborhood U of 0 in the complex plane such that $c\in U$ implies $e^{ct}\in\varrho(T(t))$. The relations (3) and (4) yield then for $c\in U$ that

$$R(c; A) = R(e^{ct}; T(t))F_c. \tag{9}$$

Here we have emphasized that $F_c=F$ depends on c, by (5), and made use of the fact that $R(e^{ct}; T(t))$ commutes with F_c.

Since (5) holds for every $x\in X$, it is easily seen that F_c is holomorphic on the whole complex plane, and (9) gives that $R(c; A)$ is holomorphic in U. Moreover, since $\lim\limits_{c\to 0}\left|\dfrac{c}{e^{ct}-1}\right|^{p+1}=t^{-p-1}>0$, there is a positive number q such that in a deleted neighborhood $U_0\subset U$ of zero

$$|c|^{p+1}\|R(c; A)\| < q|e^{ct}-1|^{p+1}\left\|R(e^{ct}; T(t))\right\|\cdot\|F_c\|. \tag{10}$$

1 is a pole of the resolvent of $T(t)$ of order p and $\|F_c\|$ is locally bounded, hence the left side of (10) converges to 0, if $c\to 0$. But then $c=0$ is a regular point or a pole of order $\leqq p$ of $R(c; A)$, and we obtain from [5, Theorem 5.8-A] that $a(A)=e(A)=m\leqq p$. Thus $A\in G_{11}$, and the proof is complete.

The following related result may also be interesting.

Theorem 4. *If $T(t)$ is of class (A) and $a(T(t)-I)=k<\infty$, then $a(A)\leqq k$.*

Proof. From (3) and (4) $AF\supset FA$, hence $F^rA^rx=V^rx$ for every $x\in D(A^r)$, $r\geqq 1$. Suppose now $x\in D(A^{k+1})$, $A^{k+1}x=0$, then $V^{k+1}x=F^{k+1}A^{k+1}x=0$. By assumption, we obtain $F^kA^kx=V^kx=0$, and we have to show that $y=A^kx=0$.

We know that $Ay=0$, and [2, Corollary 3, p. 347] yields that $T(s)y=y$ for every $s>0$, hence $y\in X_1(A)$ and, by (5), $Fy=\int\limits_0^t T(s)y\,dx=t\cdot y$. We obtain similarly that $t^ky=F^ky=F^kA^kx=0$, hence $A^kx=0$, thus $a(A)\leqq k$.

Theorem 5. *If $T(t)$ is of class $(1, A)$ and $T(t)-I\in G_2$, then $A\in G_2$.*

Proof. By assumption, $V\in G_{11}$, hence the proof of Theorem 3 yields that $a(A)=e(A)=m<\infty$. Moreover, if $m=0$, then $A\in G_1\subset G_2$, and if $m>0$, then 0 is a boundary point of $s_1(A)$. Supposing the latter, we also establish that $V\in G_4$, hence $A\in G_4$, by Corollary 1. Consequently, [4, Theorem 2.9] yields that $n(A)=d(A)<\infty$, hence $A\in G_2$.

Concerning the regularity set G_5, our result is not quite general. We shall call a projection $P\in B(X)$ an A-projection, if $P[D(A)]\subset D(A)$.

Theorem 6. *If $n(T(t)-I)<\infty$ and there is an A-projection P of X onto $R(T(t)-I)$, then $A\in G_5$.*

Proof. By assumption, with the notation $C=(I-P)X$ we have

$$X = PX\oplus(I-P)X = R(V)\oplus C.$$

Since P is an A-projection, we obtain

$$D(A) = [R(V)\cap D(A)]\oplus[C\cap D(A)], \tag{11}$$

where the members of the direct sum are closed sets in the induced topology of the subset $D(A)\subset X$. Since A is closed, $D(A)$ becomes a Banach space D under the norm

$$|x| = \|x\|+\|Ax\| \quad (x\in D).$$

It is easily seen that each set closed in the induced topology of $D(A)$ is also closed in D.

From (3) we see that $R(F)\subset D$, hence $R(V_0)=R(FA)\subset R(V)\cap D$. Further, if $y\in R(V)\cap D$, i.e. $y=Vx\in D$, then we can construct a sequence $\{x_k\}\subset D$ such that $V_0x_k\xrightarrow{D}Vx$ (here $\xrightarrow{D}$ denotes convergence in D, and $\to$ will denote convergence in X). Indeed, for $k>\omega_1$ put $x_k=kR(k;A)x\in D$, then $x_k\to x$ $(k\to\infty)$, hence $V_0x_k\to Vx$ because $V_0\subset V\in B(X)$. On the other hand $AV_0x_k=kAVR(k;A)x=kAR(k;A)y=$ $=kR(k;A)Ay\to AVx$, as asserted, hence $R(V_0)$ is D-dense in $R(V)\cap D$.

It is clear that $A\in B(D,X)$ and, since for $x\in X$ $|Fx|=\|Fx\|+\|Vx\|\leqq(K_1+K_2)\|x\|$, we establish that $F\in B(X,D)$. By assumption, $V\in B(X)$ has property (A) as defined by B. Yood [6, p. 600]: $R(V)$ is closed and $n(V)<\infty$. Since $AF=V$, [6, Theorem 3.5] yields that F has property (A). Since $V\in G_7$, Theorem 1 gives that A also has property (A). Since $V_0=FA$, we have $V_0\in B(D)$, and [6, Theorem 3.4] implies that V_0 has property (A), hence $R(V_0)$ is closed in D, consequently $R(V_0)=R(V)\cap D$.

We obtain from (11)

$$D = R(V_0)\oplus[C\cap D] \tag{12}$$

where the members of the direct sum are closed sets in D. Hence there exists a projection $Q\in B(D)$ of D onto $R(V_0)$. Since $V_0=FA$, [6, Theorem 5.1] yields that there exists a projection $R\in B(X)$ of X onto $R(A)$ and $n(A)<\infty$, thus the proof is finished.

Theorem 7. *$T(t)-I\in G_6$ implies $A\in G_6$.*

Proof. By assumption, there exists a projection Q of X onto $N(V)$; here $Q\in B(X, N(V))$. An inspection of the proof of [2, Theorem 16.7.2] yields that there always exists a projection $P\in B(N(V))$ of $N(V)$ onto $N(A)$, hence $R=PQ\in B(X)$, and the range of R is $N(A)$. For every $x\in X$ we have $R^2x=PQ(PQx)=P^2Qx=Rx$, hence R is a projection of X onto $N(A)$. Further, $V\in G_8$, thus Theorem 2 implies $A\in G_6$.

Concerning the essential spectrum s_{10} we have a positive result merely in the case $A\in B(X)$. It is nevertheless remarkable, because in general there is no containment relation between $s_{10}(f(A))$ and $f(s_{10}(A))$, if f is a complex-valued function which is locally holomorphic on an open set containing $s_1(A)$, see [1, p. 29]. (In our case $f(z)=e^{tz}$.)

Theorem 8. *Suppose $T(t)$ is a uniformly continuous group of operators, i.e. $A\in B(X)$. If $T(t)-I\in G_{10}$ for some $t\neq 0$, then $A\in G_{10}$.*

Proof. Clearly we may and will assume $t>0$. Since $A\in B(X)$, thus $V=FA$ and $F(A(X))$ is closed. Since F is continuous, the inverse image $A(X)+N(F)= =R(A)+N(F)$ is closed in X. We show that $N(F)\subset R(A)$.

Let M denote $N(V)$ and, according to the proof of [2, Theorem 16.7.2], define the projections $J_r\in B(M)$ by

$$J_r x = t^{-1}\int_0^t e^{-2\pi irs/t}\,T(s)x\,ds \quad (x\in M).$$

Then $J_r(M)=N(A-c_r)$, where $c_r=2\pi\, irt^{-1}$ (r integer). Since $s_1(A)$ is compact, there is a positive integer k such that $J_r=0$ for $|r|>k$, thus formula (16.7.5) of [2, p. 468] reduces to

$$x = \sum_{r=-k}^{k} J_r x \quad \text{for} \quad x\in M. \tag{13}$$

By (5) (with $c=0$), $Fx=\int_0^t T(s)x\,ds=tJ_0x$ for $x\in M$, thus the fact that $N(F)\subset M$ implies $N(F)=N(J_0)$. Hence for $x\in N(F)$, (13) yields

$$x = \sum_{r=-k}^{k} J_r x \quad (r\neq 0). \tag{14}$$

For $r\neq 0$ we have $J_r(M)=N(A-c_r)=\{x\in X;\, x=Ac_r^{-1}x\}\subset R(A)$, thus from (14) we obtain $N(F)\subset R(A)$, hence $A\in G_{10}$ and the proof is completed.

It is remarkable that in general no similar mapping theorem holds for the essential spectrum s_3. More exactly, we have

Theorem 9. *There is a Banach space X and an $A\in B(X)$ such that $c\in s_3(A)$ for some complex c, while $e^{ct}\notin s_3(T(t))$ for some $t>0$ (here $T(t)$ denotes the group generated by A).*

Proof. For every real number s define

$$K(s)=\frac{-s^3+(1+i)s}{s^4+3}; \tag{15}$$

K is the Fourier transform of some $k\in L_1(-\infty,\infty)$, i.e.

$$K(s)=\int_{-\infty}^{\infty} e^{ist}k(t)\,dt$$

(see, e.g., [8, pp. 13—14]). Let X denote $L_p(0,\infty)$ $(p\geqq 1)$, or any other of the spaces in (6.4) of [8, p. 38]. Define $A\in B(X)$ by

$$[Ax](t)=\int_0^{\infty} k(t-s)x(s)\,ds,$$

then $\|A\|\leqq\|k\|_1$. If z is a complex number such that $z\neq K(s)$ for $-\infty\leqq s\leqq\infty$, then

$$v=v(z)=-\frac{1}{2\pi}\int_{-\infty}^{\infty} d_s\arg\big(K(s)-z\big)=\operatorname{ind}(A-z), \tag{16}$$

moreover, $v=0$ implies $z\in\varrho(A)$, $v>0$ implies $n(A-z)=v$ and $d(A-z)=0$, while $v<0$ implies $n(A-z)=0$ and $d(A-z)=-v$ (see [8, p. 61] and [7, p. 109]).

Put $c_k=\frac{i}{8}(2k-1)$ (k integer), $r=8\pi$, then $\{c_k\}$ is the set of all complex solutions of the equation $e^{cr}=-1$. From (15) and (16) we see that because of the properties of $K(s)$

$$n(A-c_k)=\delta_{0k}\quad\text{and}\quad d(A-c_k)=\delta_{1k}, \tag{17}$$

where δ is the Kronecker symbol.

Let $T(t)$ be the group generated by A, then $T(t)$ is continuous in the uniform operator topology. [2, Theorem 16.7.2] yields that $N\big(T(r)+1\big)$ is the closed linear subspace generated by $\{N(A-c_k)\}$, hence by (17)

$$n\big(T(r)+1\big)=n(A-c_0)=1. \tag{18}$$

From (17) we see that $R(A-c_k)$ is closed, because $d(A-c_k)<\infty$ for every k. A result of Gramsch and Lay ([1, p. 22] for σ_5 and $f(z)=e^{rz}$) then yields that $d\big(T(r)+1\big)<\infty$, hence $R\big(T(r)+1\big)$ is closed. Then we have

$$n(A^*-c_k)=d(A-c_k)\quad\text{and}\quad d\big(T(r)+1\big)=n\big(T(r)^*+1\big). \tag{19}$$

$A^*\in B(X^*)$, hence it generates the uniformly continuous group $\{T(t)^*\}\subset B(X^*)$. Applying [2, Theorem 16.7.2] now to the adjoint group, we obtain from (17) and (19) that

$$d\big(T(r)+1\big)=n\big(T(r)^*+1\big)=n(A^*-c_1)=1, \tag{20}$$

hence, by (18), $\operatorname{ind}(T(r)+1)=0$. From (17) we obtain $\operatorname{ind}(A-c_0)=1$, thus $c_0\in s_3(A)$, though $e^{c_0 r}=-1\notin s_3(T(r))$. The proof is complete.

Remark. Some of the theorems and proofs obviously extend to the more general situation described in Lemma 1. Others apparently do not.

According to the results of GRAMSCH and LAY [1], if $A\in B(X)$, then some of the theorems above admit a converse in the well-known sense. However, we have

Theorem 10. *There exist a strongly continuous group $T(t)$ and a complex p such that $p\in s_i(T(1))$ for $i=1, 2, \ldots, 11$, whereas $c\in\varrho(A)$ for every complex c with $e^c=p$.*

Proof. We can take the example of [2, p. 469], and put $X=l_2$, $T(t)\{b_n\}=\{e^{int}b_n\}$. Then $T(t)$ is a strongly continuous group. It is shown there that if $p\in C\sigma(T(1))$ (the nonvoid continuous spectrum), then every c is an element of $\varrho(A)$. Moreover, with the notation $U=T(1)-p$ the set $R(U)$ is not closed, hence $p\in s_i(T(1))$ for $i=1, 2, \ldots, 10$. Since U is 1—1 and $R(U)\neq X$, we have $a(U)=0$, $e(U)\neq 0$. But then $e(U)=\infty$ (see [5, pp. 272—273]), hence $p\in s_{11}(T(1))$.

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DEPARTMENT OF MATH.
FACULTY OF CHEMICAL ENGINEERING
UNIVERSITY OF TECHNOLOGY
STOCZEK U. 2. H. II
1111 BUDAPEST