Acta Sci. Math., 38 (1976), 353—358

Note on an embedding theorem

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Let $\varphi \equiv \varphi_p$ $(p>1)$ be a nonnegative increasing function on $[0, \infty)$ with the following properties:

$$\frac{\varphi(x)}{x}\uparrow \quad \text{and} \quad \frac{\varphi(x)}{x^p}\downarrow \quad \text{as} \quad x \to \infty.$$

The set of measurable functions f on $[0, 1]$ for which $\int_0^1 \varphi(|f(x)|)\,dx < \infty$ will be denoted by $\varphi(L)$.

If $f \in \varphi(L)$, the "modulus of continuity of f with respect to φ" will be defined by

$$\omega_\varphi(\delta; f) = \sup_{0 \leqq h \leqq \delta} \bar{\varphi}\Big(\int_0^{1-h} \varphi(|f(x+h)-f(x)|)\,dx\Big) \quad (0 \leqq \delta \leqq 1),$$

where $\bar{\varphi}(x)$ denotes the inverse function of $\varphi(x)$. Given a function φ and a non-decreasing continuous function ω with $\omega(0)=0$, $H_\varphi^\omega \equiv H_\varphi^{\omega(\delta)}$ will denote the collection of functions $f(x)$ satisfying the condition

$$\omega_\varphi(\delta, f) = O(\omega(\delta)).$$

LEINDLER [2] gave a sufficient condition for $H_\varphi^{\omega(\delta)} \subset \varphi(L)\Lambda(L)$, where $\Lambda(x)$ is a "slowly increasing" function. Namely he proved the following:

Theorem A. ([2], Theorem 1) *Let $f \in \varphi(L)$ $(\varphi=\varphi_p, p \geqq 1)$ and let $\{\lambda_k\}$ be a nonnegative monotonic sequence of numbers such that*

$$\sum_{k=m}^{\infty} \frac{\lambda_k}{k^{1+\varepsilon}} \leqq K(\lambda)\frac{\lambda_m}{m^\varepsilon},^{1)}$$

Received September 5, 1975.

[1]) K and K_i denote either absolute constants or constants depending on certain functions and numbers which are not necessary to specify; $K(\alpha, \beta)$ and $K_i(\alpha, \beta)$ denote positive constants depending only on the indicated parameters. These constants are not necessarily the same at each occurrence.

where $\varepsilon=(4[p+1]+2)^{-1}$; [2]) *and let* $\Lambda(x)=\sum_{k=1}^{x}\frac{\lambda_k}{k}$. [3]) *Then*

(1) $$\sum_{n=1}^{\infty}\frac{\lambda_n}{n}\cdot\varphi\left(\omega_\varphi\left(\frac{1}{n}, f\right)\right)<\infty$$

implies $f\in\varphi(L)\Lambda(L)$ *and*

$$\int_0^1 \varphi(|f(x)|)\Lambda(|f(x)|)\,dx \leqq K(\varphi,\lambda)\left\{\sum_{n=1}^{\infty}\frac{\lambda_n}{n}\varphi\left(\omega_\varphi\left(\frac{1}{n}, f\right)\right)+\int_0^1\varphi(|f(x)|)\,dx\right\}.$$

In the present paper we are going to prove that for certain functions $\omega(\delta)$ condition (1) is also a necessary for

$$H_\varphi^{\omega(\delta)}\subset\varphi(L)\Lambda(L).$$

More precisely, we prove the following

Theorem. *Let* $\omega(\delta)$ *be a nondecreasing, continuous function with* $\omega(0)=0$, *for which the limit*

(2) $$\lim_{h\to 0}\frac{\omega\left(\frac{h}{2}\right)}{\omega(h)}\ ^{4)}$$

exists, and let $\{\lambda_k\}$ *be a nonnegative monotonic sequence of numbers satisfying* $\lambda_{k^2}\leqq K\lambda_k$ *for any k. Then a necessary and sufficient condition for*

(3) $$H_\varphi^{\omega(\delta)}\subset\varphi(L)\Lambda(L)$$

is that

(4) $$\sum_{n=1}^{\infty}\frac{\lambda_n\varphi\left(\omega\left(\frac{1}{n}\right)\right)}{n}<\infty,$$

where $\Lambda(x)$ *means the same as in Theorem A.*

1. We make use of the following:

Lemma ([3], Lemma 13). *Let* $A(u)$ *be a nonnegative nondecreasing function on* $[0,\infty)$ *such that* $A(u^2)\leqq KA(u)$ *for any* $u\in[0,\infty)$ *and let* $B(u)$ *be a nonnegative function on* $[0, 1]$. *Then*

$$\int_0^1 B(u)A(B(u))\,du<\infty \quad \textit{implies} \quad \int_0^1 B(u)A\left(\frac{1}{u}\right)du<\infty.$$

[2]) $[y]$ denotes the integral part of y.

[3]) $\sum_a^b$, where a and b are not necessarily integers, means a sum over all integers between a and b.

[4]) In the proof we shall use instead of (2) only the condition $\frac{1}{\sqrt[p]{2}}<\lim_{h\to 0}\frac{\omega(h/2)}{\omega(h)}$, where p is from the definition of the function $\varphi=\varphi_p$.

2. Proof of the Theorem

The sufficiency of (4) was proved in LEINDLER [2].

The necessity of (4) will be proved indirectly.

Suppose that

$$\sum_{n=1}^{\infty}\frac{\lambda_n\varphi\left(\omega\left(\frac{1}{n}\right)\right)}{n}=\infty. \tag{5}$$

but (3) holds. Then we can construct a function f_0 leading to a contradiction.

The construction of this function is similar to that of LEINDLER [1] made in the case $\varphi(x)=x^p$. We define $f_0(x)$ as follows:

$$f_0(x)=\begin{cases}\varrho_n, & \text{if}\quad x=3\cdot 2^{-n-2},\\ 0 & \text{if}\quad x=0,\quad x\in[\frac{1}{2},1],\quad x=2^{-n},\\ \text{linear on} & [2^{-n-1},3\cdot 2^{-n-2}],\quad [3\cdot 2^{-n-2},2^{-n}],\end{cases}$$

$(n=1,2,\ldots)$, where $\varrho_n=\bar{\varphi}\left(2^{n+1}\left(\varphi\left(\omega\left(\frac{1}{2^n}\right)\right)-\varphi\left(\omega\left(\frac{1}{2^{n+1}}\right)\right)\right)\right)$. First we show that $f_0(x)\in H_\varphi^{\omega(\delta)}$. Let

$$h\in(2^{-k-3},2^{-k-2}],\quad k\geqq 2. \tag{6}$$

Then

$$\int_0^{1-h}\varphi(|f_0(t+h)-f_0(t)|)\,dt=\left(\int_0^{3h}+\int_{3h}^{1-h}\right)\varphi(|f_0(t+h)-f_0(t)|)\,dt=I_1+I_2.$$

We have

$$I_1\leqq K(\varphi)\int_0^{4h}\varphi(|f_0(x)|)\,dx\leqq K\int_0^{2^{-k}}\varphi(|f_0(x)|)\,dx\leqq$$

$$\leqq\sum_{n=k}^{\infty}\int_{2^{-n-1}}^{2^{-n}}\varphi(|f_0(x)|)\,dx\leqq K_1\sum_{n=k}^{\infty}\varphi(\varrho_n)2^{-n-1}=$$

$$=K_1\sum_{n=k}^{\infty}\left[\varphi\left(\omega\left(\frac{1}{2^n}\right)\right)-\varphi\left(\omega\left(\frac{1}{2^{n+1}}\right)\right)\right]=K_1\varphi\left(\omega\left(\frac{1}{2^k}\right)\right)\leqq K_2\varphi(\omega(h)).$$

Next we prove that for any k:

$$\sum_{n=0}^{k}2^{-n}\varphi\left(\frac{2^n}{2^k}\bar{\varphi}\left(2^n\varphi\left(\omega\left(\frac{1}{2^n}\right)\right)\right)\right)\leqq K\varphi\left(\omega\left(\frac{1}{2^k}\right)\right). \tag{7}$$

To prove (7) we mention first of all that by (2) and (5)

$$\lim_{h\to 0}\frac{\omega\left(\frac{h}{2}\right)}{\omega(h)}=1 \tag{8}$$

follows. For, if $\lim_{h\to 0}\frac{\omega(h/2)}{\omega(h)}<q<1$, then we have

$$\varphi\left(\omega\left(\frac{1}{2^{n+1}}\right)\right)\leqq q\varphi\left(\omega\left(\frac{1}{2^n}\right)\right)$$

which by $\lambda_{k^2}\leqq K_1\lambda_k$ implies the contrary of (5).

By (8) we may assume that there exists a positive number α such that $0<\alpha<1$ and that for any $n>n_0$

$$\omega\left(\frac{1}{2^{n-1}}\right)\leqq \sqrt[p]{2}\cdot\alpha\omega\left(\frac{1}{2^n}\right). \tag{9}$$

Hence by $\varphi(kx)\leqq k^p\varphi(x)$ $(k>1)$, we have

$$\varphi\left(\omega\left(\frac{1}{2^{n-1}}\right)\right)\leqq 2\alpha^p\varphi\left(\omega\left(\frac{1}{2^n}\right)\right), \tag{10}$$

or

$$2^{n-1}\varphi\left(\omega\left(\frac{1}{2^{n-1}}\right)\right)\leqq \alpha^p 2^n\cdot\varphi\left(\omega\left(\frac{1}{2^n}\right)\right). \tag{11}$$

Since $\bar\varphi(kx)\leqq\sqrt[p]{k}\,\bar\varphi(x)$ for $k\leqq 1$ we have by (11)

$$\bar\varphi\left(2^{n-1}\varphi\left(\omega\left(\frac{1}{2^{n-1}}\right)\right)\right)\leqq\alpha\cdot\bar\varphi\left(2^n\varphi\left(\omega\left(\frac{1}{2^n}\right)\right)\right), \tag{12}$$

and consequently

$$\frac{2^{n-1}}{2^k}\bar\varphi\left(2^{n-1}\varphi\left(\omega\left(\frac{1}{2^{n-1}}\right)\right)\right)\leqq\frac{\alpha}{2}\frac{2^n}{2^k}\bar\varphi\left(2^n\varphi\left(\omega\left(\frac{1}{2^n}\right)\right)\right). \tag{13}$$

Since $\varphi(kx)\leqq k\varphi(x)$ for $k\leqq 1$, we obtain by (13),

$$\varphi\left(\frac{2^{n-1}}{2^k}\bar\varphi\left(2^{n-1}\varphi\left(\omega\left(\frac{1}{2^{n-1}}\right)\right)\right)\right)\leqq\frac{\alpha}{2}\varphi\left(\frac{2^n}{2^k}\bar\varphi\left(2^n\varphi\left(\omega\left(\frac{1}{2^n}\right)\right)\right)\right).$$

Hence,

$$2^{-n+1}\varphi\left(\frac{2^{n-1}}{2^k}\bar\varphi\left(2^{n-1}\varphi\left(\omega\left(\frac{1}{2^{n-1}}\right)\right)\right)\right)\leqq\alpha\cdot 2^{-n}\varphi\left(\frac{2^n}{2^k}\bar\varphi\left(2^n\varphi\left(\omega\left(\frac{1}{2^n}\right)\right)\right)\right),$$

which implies (7), since $0<\alpha<1$.

Having (7) we can estimate I_2. Since

$$|f_0(t+h)-f_0(t)|\leqq h\cdot 2^{n+2}(\varrho_n+\varrho_{n-1})\quad\text{if}\quad 2^{-n-1}\leqq t\leqq 2^{-n},\quad 1\leqq n\leqq k-1,$$

we have

$$I_2\leqq\int_{2^{-k}}^{2^{-1}}\varphi(|f_0(t+h)-f_0(t)|)\,dt=\sum_{n=1}^{k-1}\int_{2^{-n-1}}^{2^{-n}}\varphi(|f_0(t+h)-f_0(t)|)\,dt\leqq$$

$$\leqq K(\varphi)\sum_{n=0}^{k}2^{-n}\varphi\left(\frac{2^n}{2^k}\varrho_n\right)\leqq K_1(\varphi)\sum_{n=0}^{k}2^{-n}\varphi\left(\frac{2^n}{2^k}\bar\varphi\left(2^n\varphi\left(\omega\left(\frac{1}{2^n}\right)\right)\right)\right)\leqq$$

$$\leqq K_2(\varphi)\cdot\varphi\left(\omega\left(\frac{1}{2^k}\right)\right)\leqq K_3(\varphi)\cdot\varphi(\omega(h));$$

and hence,

$$f_0(x)\in H_\varphi^\omega.$$

Finally we prove that

$$f_0(x)\notin \varphi(L)\Lambda(L).$$

By (5)

$$\sum_{n=1}^{N}\frac{\lambda_n\varphi\left(\omega\left(\frac{1}{n}\right)\right)}{n}\to\infty\,,\quad\text{as}\quad N\to\infty. \tag{14}$$

Using (14) and $\lambda_{2n}\leqq K_1\lambda_n$, furthermore that for any N there exists an integer N_1 such that $\varphi\left(\omega\left(\frac{1}{N_1}\right)\right)\leqq\frac{1}{4K_1}\varphi\left(\omega\left(\frac{1}{N}\right)\right)$, an easy computation gives that

$$\sum_{n=1}^{\mu}\Lambda(2^n)\varphi(\varrho_n)2^{-n}\to\infty\quad\text{as}\quad\mu\to\infty. \tag{15}$$

Indeed, if $2^\mu>N_1$, we have

$$\sum_{k=1}^{N}\lambda_k k^{-1}\varphi\left(\omega\left(\frac{1}{k}\right)\right)\leqq 2\sum_{k=1}^{N}\lambda_k k^{-1}\left[\varphi\left(\omega\left(\frac{1}{k}\right)\right)-2K_1\varphi\left(\omega\left(\frac{1}{N_1}\right)\right)\right]\leqq$$

$$\leqq 2\sum_{k=1}^{2^\mu}\lambda_k k^{-1}\left[\varphi\left(\omega\left(\frac{1}{k}\right)\right)-2K_1\varphi\left(\omega\left(\frac{1}{2^\mu}\right)\right)\right]\leqq$$

$$\leqq 2\left[\sum_{n=1}^{\mu}\sum_{k=2^{n-1}+1}^{2^n}\lambda_k k^{-1}\varphi\left(\omega\left(\frac{1}{k}\right)\right)-2K_1\Lambda(2^\mu)\varphi\left(\omega\left(\frac{1}{2^\mu}\right)\right)\right]+K_2\leqq$$

$$\leqq 2\left[\sum_{n=1}^{\mu}\varphi\left(\omega\left(\frac{1}{2^{n-1}}\right)\right)\sum_{k=2^{n-1}+1}^{2^n}\lambda_k k^{-1}-2K_1\Lambda(2^\mu)\varphi\left(\omega\left(\frac{1}{2^\mu}\right)\right)\right]+K_2\leqq$$

$$\leqq 2\left[\sum_{n=2}^{\mu}2K_1\varphi\left(\omega\left(\frac{1}{2^{n-1}}\right)\right)\sum_{k=2^{n-2}+1}^{2^{n-1}}\lambda_k k^{-1}-2K_1\Lambda(2^\mu)\varphi\left(\omega\left(\frac{1}{2^\mu}\right)\right)\right]+K_3\leqq$$

$$\leqq K_4\left[\sum_{i=1}^{\mu-1}\varphi\left(\omega\left(\frac{1}{2^i}\right)\right)\sum_{k=2^{i-1}+1}^{2^i}\lambda_k k^{-1}-\Lambda(2^\mu)\varphi\left(\omega\left(\frac{1}{2^\mu}\right)\right)\right]+K_3\leqq$$

$$\leqq K_4\left[\sum_{i=1}^{\mu-1}\varphi\left(\omega\left(\frac{1}{2^i}\right)\right)\left(\Lambda(2^i)-\Lambda(2^{i-1})\right)-\Lambda(2^\mu)\varphi\left(\omega\left(\frac{1}{2^\mu}\right)\right)\right]+K_5\leqq$$

$$\leqq K_4\sum_{n=1}^{\mu-1}\Lambda(2^n)\left[\varphi\left(\omega\left(\frac{1}{2^n}\right)\right)-\varphi\left(\omega\left(\frac{1}{2^{n+1}}\right)\right)\right]+K_5\leqq$$

$$\leqq K_4\sum_{n=1}^{\mu}\Lambda(2^n)\varphi(\varrho_n)\cdot 2^{-n}+K_5,$$

which proves (15) by (14).

It is clear that for any m

$$\int_{1/2^{m+1}}^{1} \varphi(|f_0(x)|)\Lambda\left(\frac{1}{x}\right)dx = \sum_{n=0}^{m} \int_{2^{-n-1}}^{2^{-n}} \varphi(|f_0(x)|)\Lambda\left(\frac{1}{x}\right)dx \geqq$$

$$\geqq \sum_{n=0}^{m} \Lambda(2^n) \int_{2^{-n-1}}^{2^{-n}} \varphi(|f_0(x)|)\,dx \geqq K_6 \sum_{n=0}^{m} \Lambda(2^n)\varphi(\varrho_n)2^{-n},$$

and thus, by (15), we get

$$\int_0^1 \varphi(|f_0(x)|)\Lambda\left(\frac{1}{x}\right)dx = \infty. \tag{16}$$

Since $\lambda_{k^2} \leqq K_1 \lambda_k$, we have

$$\Lambda(u^2) \leqq K_2 \Lambda(u), \tag{17}$$

thus, by (16) and applying our Lemma, we obtain

$$\int_0^1 \varphi(|f_0(x)|)\Lambda(\varphi(|f_0(x)|))\,dx = \infty. \tag{18}$$

Using (17) and the properties of the function φ, we have

$$\Lambda(\varphi(x)) \leqq K_3 \Lambda(x), \tag{19}$$

whence by (18) and (19)

$$\int_0^1 \varphi(|f_0(x)|)\Lambda(|f_0(x)|)\,dx = \infty$$

follows, that is,

$$f_0 \notin \varphi(L)\Lambda(L).$$

The proof of our Theorem is completed.

References

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