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On the volume function of parallel sets

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1. Introduction

In 1959 B. SZ.-NAGY [1] proved the following theorem and its corollary:

Sz.-Nagy's Theorem. *Given an arbitrary compact set G in the plane with k connected components, if G_t denotes the parallel set of G of radius t then the function* $\text{area}(G_t)-\pi kt^2$ *is concave on* $(0, \infty)$.

Corollary. *For any bounded plane set A the function* $\text{area}\, A_t$ *is everywhere differentiable on* $(0, \infty)$ *except for a countable set of values of t. This means that the length of the parallel curves exists in the Minkowski sense for all $t>0$ outside of some countable subset of* $(0, \infty)$.

The above geometrical interpretation is based on

Pucci's Theorem. *For any subset S of Euclidean n-space E^n derivability of the function $V(t)=\text{vol}(S_t)$ at the point $r>0$ implies that the $n-1$ dimensional surface area of the boundary of S_r exists in the Minkowski sense and equals $V'(r)$.*

We remark that Sz.-Nagy's Theorem and its Corollary played a central role in proving the estimations of E. MAKAI [3] and L. E. PAYNE—H. F. WEINBERGER [4] for the foundamental frequency of planar membranes; [4] points also to the connections between Sz.-Nagy's Theorem and the isoperimetric theorem in 2 dimensions.

It is a natural problem to find generalizations of Sz.-Nagy's Theorem to higher dimensions that enables us to extend the Corollary and the results in mathematical physics mentioned above. The question is by no means trivial on account of difficulties of global differential geometrical type.

In the present paper we shall show in Theorem 1 of Section 2 that an inequality of M. KNESER [5] concerning parallel sets directly yields a simple integral representation of the volume function of parallel sets, which makes it possible to gen-

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eralize in some sense the Corollary to n dimensions and very likely opens a way of obtaining estimations concerning 3 or more dimensional vibrating bodies analogous to, but probably weaker than, those for the 2 dimensional case treated in [3] and [4].

However Theorem 1 in Section 2 does not imply the isoperimetric theorem. The main reason is the strongly local character of Kneser's inequality as shown by Lemma 5 in Section 3. Nevertheless Theorem 1 gives an idea for a new proof of less local type and a generalization of Kneser's inequality, and is suitable to extend Pucci's Theorem too. This will be the subject of Theorem 4 in Section 4 and Theorem 2 in Section 3, respectively.

2. Concavity properties of the volume function of parallel sets

Throughout this work we consider bounded subsets of E^n for an arbitrary fixed n. Let d denote the distance function[1]). Recall that the parallel set of radius t of any set A in E^n is defined by $A_t=\{p\in E^n: d(p, A)<t\}$ for $t>0$. For A fixed, the volume of A_t is a non-negative monotone increasing continuous function on $(0, \infty)$.

Our fundamental point is the following inequality

Kneser's Lemma. [5] *If* $A\subset E^n$, $b\geqq a>0$, *and* $\lambda\geqq 1$ *then*

$$\mathrm{vol}\,(A_{\lambda b}\setminus A_{\lambda a})\leqq \lambda^n\,\mathrm{vol}\,(A_b\setminus A_a).$$

(For a new proof, also applying to a more general case, see Theorem 4 in Section 4.)

Definition. We say that a continuous function f defined on some subinterval I of $(0, \infty)$ is of *Kneser type* (or a *Kneser function*) if it satisfies

$$f(\lambda b)-f(\lambda a)\leqq \lambda^n[f(b)-f(a)] \tag{1}$$

for all $a, b\in I$ with $b\geqq a$ and for $\lambda\geqq 1$.

Lemma 1. *Let f be a Kneser function on I and let a, b be two fixed points of I with $a<b$ and $f(a)\geqq f(b)$. Then the restriction of f to the interval $[b, \infty)\cap I$ is concave and monotone decreasing.*

Proof. Let $\lambda>1$, $x_0\in I$, and $x_k=\lambda^k x_0$ for $k=1, 2, \ldots$. Examine the behaviour of f restricted to the sequence $\{x_0, x_1, \ldots\}\cap I$. Let

$$\gamma_k=[f(x_k)-f(x_{k-1})]/(x_k-x_{k-1})\quad (k=1, 2, \ldots).$$

[1]) I.e. for $p, q\in E^n$ and $A\subseteq E^n$ the values $d(p, q)$ and $d(p, A)$ are the distances between the points p, q and between the point p and the set A, respectively.

Then by (1) we have

$$\gamma_{k+1} \leqq \lambda^{n-1}\gamma_k \quad (k = 1, 2, \ldots).$$

In particular, if $f(x_i) \leqq f(x_{i-1})$ holds for some i then

$$0 \geqq \gamma_i \geqq \gamma_{i+1} \geqq \gamma_{i+2} \geqq \ldots.$$

This means that the function $f|_{\{x_{-i1}, x_i, x_{i+1}, \ldots\}}$ is monotone decreasing and concave. Now let $x_0 = a$ and $\lambda = (b/a)^{2^{-m}}$ for some natural number m. Since $f(a) \geqq f(b)$, there exists at least one index i with $1 \leqq i \leqq 2^m$ for which $f(x_i) \leqq f(x_{i-1})$. Therefore with the notation

$$Q_m = \{a^{j2^{-m}} b^{k2^{-m}} : j \geqq 0,\ j+k = 2^m\} \cap I \quad (m = 1, 2, \ldots)$$

we obtain that for any m the function $f|_{Q_m}$ is monotone decreasing and concave. Since $Q_1 \subseteqq Q_2 \subseteqq \ldots$ and $\bigcup_{m=1}^{\infty} Q_m$ is dense in $[b, \infty) \cap I$, we have by the continuity of f that the statement of the lemma holds.

Lemma 2. *For any Kneser function f we have that*

(i) *f is absolutely continuous,*

(ii) *$f'(t)$ exists outside of a countable subset of* dom f,

(iii) *the left and right hand side derivatives of f ($f^{(-)}$ and $f^{(+)}$) exist at every inner point of* dom f, *and $f^{(-)} \geqq f^{(+)}$,*

(iv) *$f^{(-)}$ and $f^{(+)}$ are continuous from the left and from the right, respectively.*

Proof. Let a_0 and b_0 be arbitrarily chosen inner points of dom f with $a_0 < b_0$. Clearly, it suffices to prove that the function g defined by

$$g(t) = f(t) - t^n [f(b_0) - f(a_0)]/(b_0^n - a_0^n)$$

is concave on $[b_0, \infty) \cap \operatorname{dom} f$.

Observe that $g(a_0) = g(b_0)$ and that g also satisfies (1). Then the previous lemma shows that $g|_{[b_0, \infty)}$ is concave, which completes the proof.

Theorem 1. *If f is a function of Kneser type and $a \in \operatorname{dom} f$ then there exists a monotone decreasing function α such that*

(2) $$f(t) = \int_a^t \tau^{n-1} \alpha(\tau)\, d\tau + f(a) \quad \textit{for all} \quad t \in \operatorname{dom} f.$$

Or, which is the same, there exists a concave function $\varkappa$ such that (2) *holds with $d\varkappa(\tau)$ in place of $\alpha(\tau)\,d\tau$.*

Proof. By Lemma 2 we have $f(t) - f(a) = \int_a^t f^{(+)}(\tau)\, d\tau$. Therefore the only thing we have to prove is that the function $f^{(+)}(t) \cdot t^{1-n}$ is monotone decreasing.

Let $t\in\operatorname{dom} f$, $\lambda\geqq 1$ and $h>0$. Then (1) implies that

$$f(t+h)-f(t) \geqq \lambda^{-n}[f(\lambda t+\lambda h)-f(\lambda t)],$$

i.e.
$$[f(t+h)-f(t)]/h \geqq \lambda^{-n+1}[f(\lambda t+\lambda h)-f(\lambda t)]/(\lambda h).$$

Thus for $h\searrow 0$ we have $f^{(+)}(t)\geqq\lambda^{1-n}f^{(+)}(\lambda t)$ which estableshes $f^{(+)}(t)t^{1-n}\geqq \geqq f^{(+)}(\lambda t)(\lambda t)^{1-n}$. The proof is complete.

Remark. Relation (2) characterizes the functions of Kneser type i.e., as it can be easily seen, if any function f defined on a subinterval of $(0, \infty)$ is of the form (2), with α monotone decreasing, then f is a Kneser function.

Corollary. *For all monotone increasing Kneser functions we have*

$$f(a+\lambda y)-f(a+\lambda x) \leqq \lambda^n[f(a+y)-f(a+x)] \tag{3}$$

if $a+x$, $a+\lambda x$, $a+y$, $a+\lambda y\in\operatorname{dom} f$ *with* $a>0$, $\lambda\geqq 1$ *and* $y\geqq x\geqq 0$.

Proof. By Theorem 1 there exists a monotone decreasing function α such that

$$f(a+y)-f(a+x) = \int_{a+x}^{a+y} \tau^{n-1}\alpha(\tau)\,d\tau = \int_0^1 [\tau_1(\sigma)]^{n-1}\alpha_1(\sigma)(y-x)\,d\sigma$$

where $\tau_1(\sigma)=\sigma\cdot(a+y)+(1-\sigma)\cdot(a+x)$ and $\alpha_1(\sigma)=\alpha(\tau_1(\sigma))$.

Similarly, with the same function α,

$$f(a+\lambda y)-f(a+\lambda x) = \int_0^1 [\tau_2(\sigma)]^{n-1}\alpha_2(\sigma)\lambda\cdot(y-x)\,d\sigma$$

where $\tau_2(\sigma)=\sigma\cdot(a+\lambda y)+(1-\sigma)\cdot(a+\lambda x)$ and $\alpha_2(\sigma)=\alpha(\tau_2(\sigma))$.

Since $a, x, y\geqq 0$ and $\lambda\geqq 1$, we have $\tau_2(\sigma)\geqq\tau_1(\sigma)$ if $\sigma\in[0, 1]$. Therefore $\alpha_1(\sigma)\leqq \geqq\alpha_2(\sigma)$ for $\sigma\in[0, 1]$. But on the other hand we have $\alpha_1, \alpha_2, \lambda\cdot\tau_2\geqq 0$, consequently

$$\lambda^n[\tau_1(\sigma)]^{n-1}\alpha_1(\sigma)(y-x) \geqq [\tau_2(\sigma)]^{n-1}\alpha_2(\sigma)\cdot\lambda\cdot(y-x)$$

for all $\sigma\in[0, 1]$, which implies the statement.

Lemma 3. *Let $f_k\to f_0$ be a convergent sequence of Kneser functions defined on a common interval I. Then for any $t\in I$ we have*

$$f_0^{(-)}(t) \geqq \overline{\lim}_k f_k^{(-)}(t) \geqq \underline{\lim}_k f_k^{(+)}(t) \geqq f_0^{(+)}(t).$$

Proof. The relation $\overline{\lim}_k f_k^{(-)}(t)\geqq\underline{\lim}_k f_k^{(+)}(t)$ is trivial.

Proof of $f_0^{(-)}(t)\geqq\overline{\lim}_k f_k^{(-)}(t)$: We know that the functions

$$\alpha_k(t) = f_k^{(-)}(t)t^{1-n} \quad (k = 0, 1, \dots) \tag{4}$$

are monotone decreasing on I and satisfy

(5) $$f_k(t) = \int_a^t \tau^{n-1}\alpha_k(\tau)\,d\tau + f_k(a), \quad k = 0, 1, \dots.$$

Now assume the contrary of the statement, i.e. that for some $\varepsilon > 0$ and for a subsequence $k_1, k_2, \dots$ of subscripts we have $\lim_i \alpha_{k_i}(t) - \alpha_0(t) > \varepsilon$ for some $t \in I$. Since the left hand side derivatives of Kneser functions are continuous from the left, by the definitions of the functions α_k and since they are monotone decreasing, we obtain that there exists $\delta > 0$ such that

$$\alpha_{k_i}(\tau) \geqq \alpha_0(\tau) + \varepsilon/2 \quad \text{for} \quad \tau \in [t-\delta, t] \quad \text{and} \quad i = 1, 2, \dots.$$

Therefore for every subscript i we have

$$[f_{k_i}(t) - f_{k_i}(t-\delta)] - [f_0(t) - f_0(t-\delta)] = \int_{t-\delta}^t \tau^{n-1}[\alpha_{k_i}(\tau) - \alpha_0(\tau)]\,d\tau \geqq (\varepsilon/2) \int_{t-\delta}^t \tau^{n-1}\,d\tau =$$
$$= \text{const} > 0$$

in contradiction to the fact that $f_k \to f_0$.

The proof of $\underline{\lim}_k\, f_k^{(+)} \geqq f_0^{(+)}$ goes analogously.

Lemma 4. *Suppose that $f_1, f_2, \dots$ are Kneser functions on the domain I and suppose that the series $\sum_{k=1}^{\infty} f_k(t)$ converge for all $t \in I$. Then, if $f_0 = \sum_{k=1}^{\infty} f_k$, we have $f_0^{(+)}(t) = \sum_{k=1}^{\infty} f_k^{(+)}(t)$ and $f_0^{(-)}(t) = \sum_{k=1}^{\infty} f_k^{(-)}(t)$ for all inner points t of I.*

Remark. Since obviously f_0 is now also a Kneser function on I, the derivate numbers $f_0^{(-)}(t)$ and $f_0^{(+)}(t)$ exist for all inner points t of I.

Proof. As in the proof of Lemma 3 the functions $f_0, f_1, \dots$ can be represented in the form (5) where $\alpha_0, \alpha_1, \dots$ are defined by (4). Since the functions $\alpha_0, \alpha_1, \dots$ are monotone decreasing and continuous from the left, then if $\sum_{k=1}^{\infty} \alpha_k(t)$ also exists on I the function $\beta(t) = \sum_{k=1}^{\infty} \alpha_k(t)$ is also monotone decreasing and continuous from the left, which shows by (5) that $\beta(t) = \alpha_0(t)$ in the interior of I. Now let t be any inner point of I. By our Remark and Lemma 2 we can choose a pair of points $a, b \in I$ with $a < t < b$ where $f_0'(a)$ and $f_0'(b)$ exist. Then we have

(6) $$0 \leqq \sum_{k=1}^{m} [\alpha_k(a) - \alpha_k(t)] \leqq \sum_{k=1}^{m} [\alpha_k(a) - \alpha_k(b)] \quad (m = 1, 2, \dots).$$

On the other hand we have by Lemma 3 that $\sum_{k=1}^{\infty} \alpha_k(a)$ and $\sum_{k=1}^{\infty} \alpha_k(b)$ exist. This fact and (6) ensure the existence of $\sum_{k=1}^{\infty} \alpha_k(t)$ which completes the proof of Lemma 4.

3. An extension of Pucci's Theorem

In this and the next section we shall discuss some geometrical applications of the above results on Kneser functions. Recall that the $n-1$ dimensional Minkowski measure of any set $S\subset E^n$ is defined to equal $\lim_{t\searrow 0} \mathrm{vol}\,(S_t)/(2t)$ if this limit exists. (In the contrary case we say that S is not Minkowski measurable in $n-1$ dimensions.) We shall denote the $n-1$ dimensional Minkowski measure simply by μ.

Definition. Let X and A be subsets of E^n. We say that X is metrically associated with A if for any $p\in X$ there exists a point $q\in \bar{A}$ (the closure of A) so that $d(p, q)=$ $=d(p, A)$ and all inner points of the straight line segment joining p with q belong to X.

Remark. It is obvious that the parallel sets of a set A are metrically associated with A. Unions and intersections of sets metrically associated with A are also metrically associated with A.

Lemma 5. *Let $A\subset E^n$ and let X be a measurable set metrically associated with A. Then the function $f(t)=\mathrm{vol}\,(A_t\cap X)$ is of Kneser type.*

Remark. We can omit the proof of Lemma 5 since its statement was essentially proved by M. Kneser ([5] p. 254).

Theorem 2. *Let A be any bounded subset of E^n. Then $\mu(\partial A_t)$ [2]) esists for all $t>0$, and denoting $V(t)=\mathrm{vol}\,(A_t)$ we have*

$$\mu(\partial A_t) = \frac{1}{2}[V^{(-)}(t)+V^{(+)}(t)].$$

Proof. It is enough to consider the case $t=1$ i.e. it suffices to see that

$$\mu(\partial A_1) = \frac{1}{2}[V^{(-)}(1)+V^{(+)}(1)].$$

Introduce the extended real valued function $h\colon E^n\to[0, \infty]$ which is defined as follows: For any point $x\in E^n$ let $h(x)$ be the least upper bound of all numbers l for which there exist points $p\in\bar{A}$ and $q\in E^n$ such that $l=d(p, q)=d(q, A)$ and the point x lies on the closed straight line segment joining p with q.

It follows directly from this definition that the inverse images $h^{-1}(a)$ for any $a\in[-\infty, \infty]$ are metrically associated with A. Furthermore, it is easy to observe that the sets $h^{-1}([a, \infty])$ are closed, and therefore if B is any Borel subset of $[-\infty, \infty]$ then $h^{-1}(B)$ is measurable and metrically associated with A.

Let us define the following functions on $(0, \infty)$:

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[2]) For any set $S\subseteq E^n$ the symbol ∂S denotes its boundary.

For any Borel subset B of $[-\infty, \infty]$ let V_B be the function

$$V_B(t) = \operatorname{vol}\left(A_t \cap h^{-1}(B)\right).$$

Now by Lemma 5 we have that all the functions V_B are of Kneser type.

Next, let us examine the behavior of vol $((\partial A_1)_t)$ for $t \searrow 0$.

It is well-known that the sets $(\partial A_1)_t$ can be represented in the form

$$(\partial A_1)_t = [A_{1+t} \setminus \overline{A_{1-t}}] \setminus Y(t) \quad (t \in (0, 1))$$

where

$$Y(t) = \{p : 1 > d(p, A) > 1-t \quad \text{and} \quad d(p, \partial A_1) > t\}.$$

By Lemma 2 the only thing we have to prove is that

$$\lim_{t \searrow 0} t^{-1} \operatorname{vol}\left(Y(t)\right) = 0. \tag{6}$$

For this we only need to observe that

$$Y(t) \subseteqq h^{-1}([0, 1)) \cap (A_1 \setminus A_{1-t}) \quad \text{for} \quad t \in (0, 1). \tag{7}$$

The inclusions $Y(t) \subseteqq A_1 \setminus A_{1-t}$ are obvious. Now suppose that for some point $x \in Y(t)$ we have $h(x) \geqq 1$. This means by definition of $h(x)$ that for some $q \in E^n$ and $p \in \bar{A}$ the point x lies on the closed segment between p and q and $d(p, q) = d(q, A) \geqq 1$ holds. Therefore there is a point $\tilde{q}$ on the closed segment pq lying at a distance 1 from p, and we have

$$1 = d(\tilde{q}, A) = d(\tilde{q}, p) \tag{8}$$

$$d(\tilde{q}, x) = d(\tilde{q}, p) - d(x, p) = 1 - d(x, A) \leqq t. \tag{9}$$

But (9) contradicts the fact implied by (8) that $\tilde{q} \in \partial A_1$, since by $x \in Y(t)$ we have $d(x, \partial A_1) > t$. Thus we have proved (7).

By (7) we have

$$0 \leqq \operatorname{vol}\left(Y(t)\right) \leqq V_{[0,1)}(1) - V_{[0,1)}(1-t) = \sum_{k=1}^{\infty} \left\{ V_{\left[1-\frac{1}{k}, 1-\frac{1}{k+1}\right)}(1) - V_{\left[1-\frac{1}{k}, 1-\frac{1}{k+1}\right)}(1-t) \right\}.$$

Consequently, by Lemma 4,

$$0 \leqq \lim_{t \searrow 0} t^{-1} \operatorname{vol}\left(Y(t)\right) \leqq \sum_{k=1}^{\infty} V^{(-)}_{\left[1-\frac{1}{k}, 1-\frac{1}{k+1}\right)}(1) \tag{10}$$

holds. However, any function $V_{[a, b)}$ is constant for $t > b$, therefore the right hand side of inequality (10) equals 0 which proves (6) and the theorem itself.

Beside this generalization of Pucci's Theorem we mention here as a consequence of Section 2 concerning the Minkowski measurability of the boundary of parallel sets the following approximation theorem:

Theorem 3. *Let* $\{A^k\}_{k=1}^{\infty}$ *be a sequence of non-empty bounded subsets of* E^n *tending in Hausdorff distance to a bounded set* A_0.[3]) *Then the relation* $\lim_k \mu(\partial A_t^k) = \mu(\partial A_t^0)$ *holds for all* $t \in (0, \infty)$ *except for a countable subset of* $(0, \infty)$.

[3]) The Hausdorff distance between $X, Y \subseteqq E^n$ is defined by $\inf\{\delta > 0 : X \subseteqq Y_\delta \text{ and } Y \subseteqq X_\delta\}$.

Proof. For $k=0, 1, 2, \ldots$ let $V_k(t)$ denote the volume function of the parallel sets of the set A^k and let ε_k be the Hausdorff distance of A^k from A^0. Since obviously $A^0_{t-(\varepsilon_k+1/k)} \subseteqq A^k_t \subseteqq A^0_{t+(\varepsilon_k+1/k)}$ whenever $t > \varepsilon_k + 1/k$, by the continuity of V_0 we have

$$V_k(t) \to V_0(t) \quad \text{for} \quad t > 0 \quad \text{and} \quad k \to \infty.$$

Then Lemma 3 implies that for all points t where $V'(t)$ exists,

$$\mu(\partial A^k_t) = \frac{1}{2}[V_k^{(-)}(t) + V_k^{(+)}(t)] \to V_0'(t) = \mu(\partial A^0_t)$$

holds if $k \to \infty$ which completes the proof.

4. A new proof and a generalization of Kneser's Lemma

Theorem 1 has a simple geometrical interpretation which enables us to give a new proof to Kneser's Lemma.

Let A be an arbitrary bounded subset of E^n and let $f(t) = \mathrm{vol}\,(A_t)$. We have to prove that f is a Kneser function.

Observe that it suffices to prove Kneser's Lemma for sets A consisting of merely finitely many points, since the general case can be obtained from here by the following simple approximation procedure: Choose any countable subset $\{p_1, p_2, \ldots\}$ of A, dense in A, and take the functions $f_k(t) = \mathrm{vol}\,(\{p_1, p_2, \ldots, p_k\}_t)$ $(k=1, 2, \ldots)$. Since obviously $f_k \to f$ for $k \to \infty$, we have that if $f_1, f_2, \ldots$ are functions of Kneser type then so is f too.

Thus let $A = \{p_1, \ldots, p_k\}$. In order to simplify the notations, we consider throughout this section a fixed point z as the origin of E^n and all the points p of the space E^n will be identified with the vector of the directed line segment $\overrightarrow{zp}$. Further let K^0 denote the open unit ball of centre z in E^n.

Then A_t can be written in the form of the following Minkowski sum:

$$A_t = A + tK^0 = \bigcup_{i=1}^{k} (p_i + tK^0) = \bigcup_{i=1}^{k} [D_i \cap (p_i + tK^0)] \tag{11}$$

where D_i denotes the Dirichlet cell of p_i with respect to $\langle p_1, \ldots, p_n \rangle$ i.e.

$$D_i = \{p : d(p, p_j) \geqq d(p, p_i) \quad \text{if} \quad j \leqq i \quad \text{and} \quad d(p, p_j) > d(p, p_i) \quad \text{if} \quad j > i\}$$
$$(i = 1, 2, \ldots, k).$$

Since $D_1, \ldots, D_k$ are pairwise disjoint convex figures (not necessarily bounded polyhedra), (11) implies that

$$\mathrm{vol}\,(A_t) = \sum_{i=1}^{k} \mathrm{vol}\,[D_i \cap (p_i + tK^0)] = \sum_{i=1}^{k} \int_0^t \mathrm{area}\,[D_i \cap \partial(p_i + \tau K^0)]\,d\tau. \tag{12}$$

Observe that any cell D_i is starshaped from the point p_i (implied by convexity of D_i), and therefore the angles consisting of the rays issued from p_i and joining p_i with the points of the figure $D_i \cap \partial(p_i+tK^0)$ on the sphere give a monotone decreasing set valued function of the variable t. Consequently, the functions α_i defined by

$$\alpha_i(t) = t^{1-n} \operatorname{area} [D_i \cap \partial(p_i+tK^0)] \quad (i = 1, \dots, k)$$

are monotone decreasing. Thus for $\alpha(\tau)=\sum_{i=1}^{k} \alpha_i(\tau)$ we have by (12) that $f(t)= =\operatorname{vol}(A_t)=\int_0^t \tau^{n-1}\alpha(\tau)\,d\tau$ which means that f is a Kneser function. Qu.e.d.

The application of Dirichlet cells enables us to extend Kneser's Lemma as follows:

Theorem 4. *Let K be an arbitrary open bounded central symmetrical convex figure of E^n and let $A \subset E^n$ be also bounded. Then the function $V(t)=\operatorname{vol}(A+tK)$ is of Kneser type.*

Proof. It is easy to see that it suffices to restrict our attention to the case of $A=\{p_1, \dots, p_k\}$ as above. We may assume without any loss of generality that z is the centre of K. Introduce the function $\varrho: E^n \times E^n \to [0, \infty)$ defined as follows: For $x, y \in E^n$ let $\varrho(x, y)$ be equal to the unique coefficient σ for which the inclusion $y \in \partial(x+\sigma K)$ holds.

Since now we have that $(-1)K=K$, the function ϱ will be a translation invariant metric on E^n, i.e.

(13) $\varrho(x, y) = 0$ if and only if $x = y$,

(14) $\varrho(x, y)+\varrho(y, u) \geqq \varrho(x, u)$,

(15) $\varrho(x, y)+\varrho(y, u) = \varrho(x, u)$ if y belongs to the closed segment xu.

In this case it is convenient to consider

$$D_i = \{p : \varrho(p_i, p) \leqq \varrho(p, p_j) \text{ if } j \leqq i \text{ and } \varrho(p_j, p) > \varrho(p, p_i) \text{ if } j > i\}$$

$(i=1, \dots, k)$. Then for the same reason as by which (12) was obtained we have

$$V(t) = \sum_{i=1}^{k} \operatorname{vol} [D_i \cap (p_i+tK)].$$

On the other hand, one can prove that any figure D_i is starshaped with respect to the point p_i.

In fact. Fix an arbitrary index i, and let $p \in D_i$, $\beta \in [0, 1]$, and $q=p_i+\beta \cdot (p-p_i)$. We have to point out that $q \in D_i$, i.e.

$$\varrho(q, p_j) \geqq \varrho(p_i, q) \quad \text{if} \quad j \leqq i \tag{16}$$

$$\varrho(p_j, q) > \varrho(q, p_i) \quad \text{if} \quad j > i. \tag{17}$$

Let e.g. $j \leqq i$. Then by (14) and (15) we have

$$\varrho(q, p)+\varrho(p, p_j) \geqq \varrho(q, p_j), \tag{18}$$

$$\varrho(p_i, q)+\varrho(q, p) = \varrho(p_i, p). \tag{19}$$

By the definition of D_i, relation $p \in D_i$ implies that

$$\varrho(p_j, p) \geqq \varrho(p_i, p). \tag{20}$$

But (18), (19) and (20) immediately yield (16). The way to obtain (17) is similar.

Now the fact that D_i is a starshaped domain with respect to p_i can be formulated in terms of Minkowski sums as

$$(1-\beta) \cdot p_i + \beta D_i \subseteqq D_i \quad \text{for any} \quad \beta \in [0, 1]. \tag{21}$$

From here it easily follows that the function $f(t) = \mathrm{vol}\,[D_i \cap (p_i + tK)]$ is of Kneser type. In order to prove this let $b \geqq a \geqq 0$ and $\lambda \geqq L$. We have to see that

$$\mathrm{vol}\,[D_i \cap \{p_i + (\lambda bK \setminus \lambda aK)\}] \leqq \lambda^n \,\mathrm{vol}\,[D_i \cap \{p_i + (bK \setminus aK)\}].$$

For this it suffices to prove that the homothetic image of the set $D_i \cap \{p_i + (\lambda bK \setminus \lambda aK)\}$ from the point p_i with coefficient λ^{-1} is included in $D_i \cap \{p_i + (bK \setminus aK)\}$. Or which is the same, we have to prove

$$[\beta D_i + (1-\beta) p_i] \cap \{p_i + (bK \setminus aK) \subseteqq D_i \cap \{p_i + (bK \setminus aK)\}$$

for $\beta = \lambda^{-1} (\in [0, 1])$. But this is a direct corollary of (21).

Remark. It is not hard to see that no analogue of Lemma 5 holds in this generality if we replace A_t by $A + tK$ where K denotes a central symmetrical convex figure and if we replace the metric d of E^n by the metric ϱ defined in the above proof in terms of K. This fact clearly shows the essential differences between the original and the present proof of Kneser's Lemma.

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