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Unitary subsemigroups in commutative semigroups

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1. Introduction. We use the terminology and notations of [1]. In particular, a subset U of a semigroup S will be called *left [right] unitary* if, for each $u \in U$ and $s \in S$, $us \in U$ [$su \in U$] implies $s \in U$; a subset which is both left and right unitary will be called *unitary*.

In this paper we deal only with commutative semigroups. Clearly, the terms "left unitary", "right unitary" and "unitary" have the same meaning in this case.

2. Connections with a special congruence relation. Let S be a commutative semigroup and R a subsemigroup of S. Define $a\varrho_R b$ $(a, b \in S)$ to mean that there exists an $x \in R$ such that $ax = bx$. It is well-known that ϱ_R is a congruence on S. T. TAMURA and H. B. HAMILTON discussed in [4] the case when R is cofinal in S (that is, to each $s \in S$ there exists an $r \in R$ such that $sr \in R$). A part of their results can be formulated as follows: *If R is a cofinal subsemigroup of the commutative semigroup S, then*

(i) *R is included in a ϱ_R-class (i.e., $x\varrho_R y$ for each $x, y \in R$), but*

(ii) *R is itself a ϱ_R-class if and only if it is unitary.*

Now we show that (i) and (ii) remain true if cofinality is replaced by the condition that R is a subsemilattice of S. We recall that a *semilattice* is a commutative semigroup every element of which is idempotent.

Theorem 1. *Let S be a commutative semigroup and R a subsemilattice of S. Then $x\varrho_R y$ for each pair $x, y \in R$.*

Proof. For any elements x, y of R we have $x \cdot xy = y \cdot xy$ and $xy \in R$. Hence $x\varrho_R y$ indeed.

Before formulating the analogue of (ii) we prove a more general proposition:

Theorem 2. *Let S be a commutative semigroup and R a unitary subsemigroup in S. Then $u\varrho_R a$ $(a \in R)$ implies $u \in R$ (i.e., R is the union of some ϱ_R-classes).*

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Proof. Let $a\in R$, $u\in S$ and $u\varrho_R a$. Then there exists an $x\in R$ such that $xu=xa\in R$. Since R is unitary, $u\in R$.

Theorem 3. *Let S be a commutative semigroup and R a subsemilattice of S. Then R is a ϱ_R-class if and only if it is unitary.*

Proof. If R is unitary, then it is a ϱ_R-class by Theorems 1 and 2. Conversely, suppose that R is a ϱ_R-class and $ax=b$ with $a, b\in R$. Then $ax=a^2x=ab$ and therefore $x\varrho_R b$. Since R is a ϱ_R-class, we conclude that $x\in R$. This means that R is unitary, indeed.

3. Unitary subsemilattices in semilattices. A subsemilattice F of a semilattice S is called a *filter* if, for any elements $e\in F$ and $s\in S$, $es=e$ implies $s\in F$. By the following theorem the filters and the unitary subalgebras will be identified in semilattices:

Theorem 4. *The following assertions concerning a subsemilattice R of a semilattice S are equivalent:*

(A) *R is a ϱ_R-class;*

(B) *R is a filter;*

(C) *R is unitary.*

Proof. Since (A) and (C) are equivalent by Theorem 3, we have only to show that (B) and (C) are also equivalent.

Let $ax=b$ with $a, b\in R$. Then $b=ax^2=bx$. Assuming (B), we get $x\in R$. This means that (B) implies (C).

Let $a=as$ with $a\in R$, $s\in S$. Assuming (C), we get $s\in R$. This means that (C) implies (B), too.

In the rest of this paper we point to a prominent role of unitary subsemilattices. Let S and Σ be semilattices with identity elements e and ε, respectively. Let, further, a^b $(a, b\in S)$ denote a mapping of $S\times S$ into Σ. Define a multiplication in $S\times\Sigma$ by the rule

$$(a,\alpha)\circ(b,\beta)=(ab,\, a^b\alpha\beta). \tag{1}$$

The resulting grupoid, denoted by $S\circ\Sigma$, is a (degenerated) Rédeian skew product of S and Σ in the sense of [2]. It was shown in [3] that $S\circ\Sigma$ is a semilattice if and only if $a^b=b^a$ and

$$a^a=\varepsilon \tag{2}$$

for each $a, b\in S$. Now we prove

Theorem 5. *Let S and Σ be semilattices with the identity elements e and ε, respectively. If their Rédeian skew product $S\circ\Sigma$ is a semilattice, too, then the set*

$$\Gamma = \{(e,\alpha)\colon\ \alpha\in\Sigma\}$$

is a subsemilattice of $S\circ\Sigma$ such that

(i) *Γ is unitary and isomorphic with Σ;*
(ii) *$S\circ\Sigma/\varrho_\Gamma$ is isomorphic with S.*

Proof. By (1), Γ is a subalgebra of $S\circ\Sigma$. Property (i) can be derived immediately from (1) and (2). As for (ii), $(a,\alpha)\varrho_\Gamma(b,\beta)$ means that there exists an (e,γ) such that $(a,\alpha)\circ(e,\gamma)=(b,\beta)\circ(e,\gamma)$ which implies $a=b$. Conversely, $a=b$ implies $(a,\alpha)\varrho_\Gamma(b,\beta)$ for arbitrary $\alpha,\beta\in\Sigma$ because $(a,\alpha)\circ(e,\alpha\beta)=(a,a^e\alpha\beta)=(b,b^e\alpha\beta)=$ $=(b,\beta)\circ(e,\alpha\beta)$ in this case. Thus (ii) is proved, too.

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