Acta Sci. Math., 39 (1977), 3—14

On intertwining dilations. II

T. ANDO, Z. CEAUŞESCU and C. FOIAŞ

1. In this paper we shall consider only (linear bounded) operators on (either all real, or all complex) Hilbert spaces. As usual, $L(\mathfrak{H}', \mathfrak{H})$ will denote the space of all operators from $\mathfrak{H}'$ into $\mathfrak{H}$ and by $L(\mathfrak{H})$ the space $L(\mathfrak{H}, \mathfrak{H})$. Let $T_i \in L(\mathfrak{H}_i)$ be a contraction; and let $U_i \in L(\mathfrak{K}_i)$ be its minimal isometric dilation $(i=1, 2)$. Also, let us denote by $I(T_1; T_2)$ the set of all operators $A \in L(\mathfrak{H}_2, \mathfrak{H}_1)$ intertwining T_1 and T_2 (i.e. $T_1 A = A T_2$). By an *exact intertwining dilation* (EID) of $A \in I(T_1; T_2)$ we mean any $B \in L(\mathfrak{K}_2, \mathfrak{K}_1)$ satisfying

$$(1.1) \qquad P_{\mathfrak{H}_1} B = A P_{\mathfrak{H}_2}, \quad B \in I(U_1; U_2) \quad \text{and} \quad \|B\| = \|A\|,$$

(where $P_{\mathfrak{H}_i}$ is the orthogonal projection of $\mathfrak{K}_i$ onto $\mathfrak{H}_i$ $(i=1, 2)$).

In order to state our sufficient and necessary conditions for the uniqueness of the EID of a contraction $\in I(T_1; T_2)$ we also need the concept of the regularity of a factorization of a contraction as a product of two contractions (see [9], Ch. VII, § 3 and [10]). Namely, for two contractions $A_1 \in L(\mathfrak{A}, \mathfrak{B})$, $A_2 \in L(\mathfrak{B}, \mathfrak{A}_*)$ the factorization of $A_2 A_1 \in L(\mathfrak{A}, \mathfrak{A}_*)$ as the product of A_2 and A_1 is called *regular* if

$$(1.2) \qquad \{D_{A_2} A_1 a \oplus D_{A_1} a : a \in \mathfrak{A}\}^- = (D_{A_2} \mathfrak{B})^- \oplus (D_{A_1} \mathfrak{A})^-,$$

where, as usual, for any contraction C, D_C denotes the defect operator $(1 - C^* C)^{1/2}$.

Our main result which was suggested by [1], [2] and [3] is given by the following

Theorem 1.1. *Let $A \in L(\mathfrak{H}_2, \mathfrak{H}_1)$, $\|A\| = 1$, intertwine the contractions T_1 and T_2. A sufficient and necessary condition for A to have a unique exact intertwining dilation is that at least one of the factorizations $A \cdot T_2$ or $T_1 \cdot A$ (of $A T_2 = T_1 A$) be regular.*

Received June 25, 1976.

The research of the first author was supported by the Hungarian Institute of Cultural Relations and the Japan Society for the Promotion of Science.

The next three sections are devoted to the proof of this theorem. Some complements and connections with results of [1], [2], [3] and [5] will be discussed in sections 5 and 6.

The authors take this opportunity to express their thanks to Prof. B. Sz.-Nagy for his stimulating interest in this research.

2. Let us start with some simple preliminaries. For a contraction $T_i \in L(\mathfrak{H}_i)$ we denote, as above, by $U_i \in L(\mathfrak{K}_i)$ its minimal isometric dilation; and we shall denote by $\hat{U}_i \in L(\hat{\mathfrak{K}}_i)$ the minimal unitary dilation of U_i, which is also the minimal unitary dilation of T_i $(i=1, 2)$.

By the construction of $\hat{U}_i$ (see [9], Ch. I and II) it is known that $\hat{U}_i$ is the minimal unitary dilation and $U_i^{(*)} = \hat{U}_i^{-1} | \mathfrak{K}_i^{(*)}$ is the minimal isometric dilation, of T_i^*, where

$$\mathfrak{K}_i^{(*)} = \hat{\mathfrak{K}}_i \ominus \bigvee_{n=0}^{\infty} U_i^n \mathfrak{L}_i \quad \text{and} \quad \mathfrak{L}_i = ((U_i - T_i)\mathfrak{H}_i)^- \quad (i = 1, 2).$$

Also, it is well known that any EID B of A has a unique extension $\hat{B} \in L(\hat{\mathfrak{K}}_2, \hat{\mathfrak{K}}_1)$ satisfying: $\hat{B}\hat{U}_2 = \hat{U}_1 \hat{B}$, $\|\hat{B}\| = \|A\|$ and $\hat{P}_{\mathfrak{H}_1} \hat{B} | \mathfrak{H}_2 = A$, where $\hat{P}_{\mathfrak{H}_1}$ denotes the orthogonal projection of $\hat{\mathfrak{K}}_1$ onto $\mathfrak{H}_1$ ([9], Ch. II, § 2). Now, it is easy to see that if $B_* \in I(U_2^{(*)}; U_1^{(*)})$ is an EID of $A^* \in I(T_2^*; T_1^*)$ then $(\hat{B}_*)^* | \mathfrak{K}_2$ is an EID of A, and conversely, if $B \in I(U_1; U_2)$ is an EID of $A \in I(T_1; T_2)$ then $(\hat{B})^* | \mathfrak{K}_1^*$ is an EID of A^*. So we can conclude with the following

Lemma 2.1. *$A \in I(T_1; T_2)$ has a unique EID if and only if $A^* \in I(T_2^*; T_1^*)$ has a unique EID.*

Another simple fact is condensed in the following

Remark 2.1. With the above notations, let $A \in I(T_1; T_2)$ be a contraction and let $\tilde{A} = AP_{\mathfrak{H}_2}$. Plainly, $\tilde{A} \in I(T_1; U_2)$; and any EID of $\tilde{A}$ is an EID of A and vice-versa (see [9], Ch. II, § 2). Consequently, A has a unique EID if and only if $\tilde{A}$ enjoys the same property.

Finally, in the sequel we shall also use the following

Lemma 2.2. *Let $A \in L(\mathfrak{A}, \mathfrak{B})$, $T \in L(\mathfrak{A})$ be contractions and U the minimal isometric dilation of T on $\tilde{\mathfrak{A}} = \bigvee_{n=0}^{\infty} U^n \mathfrak{A}$. Let $\tilde{A} = AP \in L(\tilde{\mathfrak{A}}, \mathfrak{B})$, where P is the orthogonal projection of $\tilde{\mathfrak{A}}$ onto $\mathfrak{A}$. Then, the factorization $\tilde{A} \cdot U$ of $\tilde{A}U$ is regular if and only if so is the factorization $A \cdot T$ of AT.*

Proof. Let us first observe that

$$\begin{aligned} \text{(2.1)} \qquad \|D_{\tilde{A}}(\tilde{a} - U\tilde{a}')\|^2 &= \|\tilde{a} - U\tilde{a}'\|^2 - \|AP(\tilde{a} - U\tilde{a}')\|^2 = \\ &= \|D_A P(\tilde{a} - U\tilde{a}')\|^2 + \|(I-P)(\tilde{a} - U\tilde{a}')\|^2 = \\ &= \|D_A(P\tilde{a} - TP\tilde{a}')\|^2 + \|(I-P)(\tilde{a} - U\tilde{a}')\|^2, \end{aligned}$$

for all $\tilde{a}, \tilde{a}' \in \tilde{\mathfrak{A}}$. Now, let us assume that the factorization $\tilde{A} \cdot U$ of $\tilde{A}U$ is regular, i.e.

$$(D_{\tilde{A}} U \tilde{\mathfrak{A}})^- = (D_{\tilde{A}} \tilde{\mathfrak{A}})^-. \tag{2.2}$$

For any $a, a' \in \mathfrak{A}$, we consider

$$\tilde{a} = a + (U-T)a' \in \tilde{\mathfrak{A}}. \tag{2.3}$$

Then, from (2.2) it follows that there exists a sequence $(\tilde{a}_j)_{j=1}^{\infty} \subset \tilde{\mathfrak{A}}$ such that

$$\|D_{\tilde{A}}(\tilde{a} - U\tilde{a}_j)\| \to 0 \quad (j \to \infty). \tag{2.4}$$

Also, for $\tilde{a}$ and $\tilde{a}_j$ satisfying (2.3) and (2.4), we have, by (2.1)

$$\begin{aligned}\|D_{\tilde{A}}(\tilde{a} - U\tilde{a}_j)\|^2 &= \|D_A(a - TP\tilde{a}_j)\|^2 + \|(U-T)a' - (I-P)U\tilde{a}_j\|^2 = \\ &= \|D_A(a - TP\tilde{a}_j)\|^2 + \|(U-T)(a' - P\tilde{a}_j)\|^2 + \|(I-P)U(I-P)\tilde{a}_j\|^2 = \\ &= \|D_A(a - TP\tilde{a}_j)\|^2 + \|D_T(a' - P\tilde{a}_j)\|^2 + \|(I-P)\tilde{a}_j\|^2.\end{aligned}$$

From this and from (2.4) we infer that

$$\{D_A Ta \oplus D_T a : a \in \mathfrak{A}\}^- = (D_A \mathfrak{A})^- \oplus (D_T \mathfrak{A})^-, \tag{2.5}$$

i.e., the factorization $A \cdot T$ of AT is regular. Conversely, let us assume that (2.5) holds. Hence, for any $a, a' \in \mathfrak{A}$ there exists $(a_j)_{j=1}^{\infty} \subset \mathfrak{A}$ such that

$$\|D_A(a - Ta_j)\|^2 + \|D_T(a' - a_j)\|^2 \to 0 \quad (j \to \infty). \tag{2.6}$$

Then, for any $\tilde{a} \in \tilde{\mathfrak{A}}$ of the form

$$\tilde{a} = a + (U-T)a' + \tilde{a}'', \tag{2.7}$$

where $a, a' \in \mathfrak{A}$ and $\tilde{a}'' \in U(I-P)\tilde{\mathfrak{A}}$, consider the elements

$$\tilde{a}_j = a_j + U^* \tilde{a}'' \in \tilde{\mathfrak{A}} \quad (j = 1, 2, \ldots), \tag{2.8}$$

where $(a_j)_{j=1}^{\infty} \subset \mathfrak{A}$ is the sequence occurring in (2.6). By virtue of (2.1) we have for $\tilde{a}$ and $\tilde{a}_j$ given in (2.7) and (2.8)

$$\begin{aligned}\|D_{\tilde{A}}(\tilde{a} - U\tilde{a}_j)\|^2 &= \|D_A(a - Ta_j)\|^2 + \|(U-T)a' + \tilde{a}'' - (I-P)U\tilde{a}_j\|^2 = \\ &= \|D_A(a - Ta_j)\|^2 + \|(U-T)(a' - a_j)\|^2 + \|\tilde{a}'' - (I-P)UU^*\tilde{a}''\|^2 = \\ &= \|D_A(a - Ta_j)\|^2 + \|D_T(a' - a_j)\|^2.\end{aligned}$$

Thus, from (2.6), it follows that $D_{\tilde{A}}\tilde{a} \in (D_{\tilde{A}} U \tilde{\mathfrak{A}})^-$, for any $\tilde{a}$ of the form (2.7). Since the set of these $\tilde{a}$ is dense in $\tilde{\mathfrak{A}}$, (2.2) follows at once.

Remark 2.2. In the sequel we shall also use the following characterization of regular factorization. Namely, (1.2) is equivalent to any one of the relations

$$D_{A_2} \mathfrak{B} \cap D_{A_1^*} \mathfrak{B} = \{0\}, \tag{2.9}$$

$$D_{A_2} \mathfrak{B} \cap \ker A_1^* = \{0\} \quad \text{and} \quad D_{A_1} \mathfrak{A} \cap A_1^* D_{A_2} \mathfrak{B} = \{0\}. \tag{2.10}$$

For the equivalence of (1.2) and (2.9) we refer to [6] and [10]. On the other hand, if (2.9) holds then the first relation of (2.10) follows from the inclusion $\ker A_1^* \subset D_{A_1^*}\mathfrak{B}$ while if $D_{A_1}a = A_1^* b$ for some $b \in D_{A_2}\mathfrak{B}$ then by virtue of the relation $A_1 D_{A_1} = D_{A_1^*} A_1$ we have

$$b = D_{A_1^*}^2 b + A_1 A_1^* b = D_{A_1^*}(D_{A_1^*} b + A_1 a),$$

hence $b=0$. Thus (2.9) implies (2.10). Conversely if (2.10) holds and if $D_{A_2} b = D_{A_1^*} b'$ for some $b, b' \in \mathfrak{B}$, then $A_1^* D_{A_2} b = D_{A_1} A_1^* b'$, therefore $D_{A_2} b = 0$, i.e. (2.9) holds too.

Remark 2.3. Let $A \in L(\mathfrak{A}, \mathfrak{B})$, $\tilde{A} \in L(\tilde{\mathfrak{A}}, \mathfrak{B})$ be as in Lemma 2.2 and let $T' \in L(\mathfrak{B})$ be a contraction. Then, since $D_{\tilde{A}^*} = D_{A^*}$, it is obvious (by virtue of the preceding remark) that the factorization $T' \cdot \tilde{A}$ of $T'\tilde{A}$ is regular if and only if so is the factorization $T' \cdot A$ of $T'A$.

3. In order to prove the sufficiency of the condition in Theorem 1.1, we shall firstly consider the case when T_2 is an isometry. For the simplification of the notations, we shall introduce the following notations: $\mathfrak{H}_1 = \mathfrak{H}$, $T_1 = T$, $U \in L(\mathfrak{K})$ — the minimal isometric dilation of T, and $\mathfrak{H}_2 = \mathfrak{G}$, $T_2 = Z$.

Let us also denote by $P_{(n)}$ the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}_{(n)} = \mathfrak{H} \oplus \mathfrak{L} \oplus \ldots \oplus U^{n-1}\mathfrak{L}$, where $\mathfrak{L} = ((U-T)\mathfrak{H})^-$, $P_{(0)} = P_{\mathfrak{H}}$, and $T_{(n)} = P_{(n)} U | P_{(n)}\mathfrak{K}$ $(n = 1, 2, \ldots)$, $T_{(0)} = T$; also for any $A \in I(T; Z)$, $\|A\| = 1$, let us set

$$\mathscr{B}_{T_{(1)}}(A) = \{B_1 \in L(\mathfrak{G}, \mathfrak{H}_{(1)}) : T_{(1)} B_1 = B_1 Z,\ \|B_1\| = 1,\ P_{\mathfrak{H}} B_1 \doteq A\}. \tag{3.1}$$

In order to show that $\mathscr{B}_{T_{(1)}}(A)$ is not empty we recall the first step of the construction of an EID of A (see [9], Ch. II, § 2). We have to determine an operator of the form

$$B_1 = \begin{bmatrix} A \\ X \end{bmatrix} : \mathfrak{G} \to \mathfrak{H}_{(1)} = \begin{matrix} \mathfrak{H} \\ \oplus \\ \mathfrak{L} \end{matrix} \tag{3.2}$$

satisfying the conditions

$$\|Xg\| \leqq \|D_A g\| \quad (g \in \mathfrak{G}), \tag{3.3}$$

$$T_{(1)} B_1 = B_1 Z, \tag{3.4}$$

where

$$T_{(1)} = \begin{bmatrix} T & 0 \\ U-T & 0 \end{bmatrix} : \begin{matrix} \mathfrak{H} \\ \oplus \\ \mathfrak{L} \end{matrix} \to \begin{matrix} \mathfrak{H} \\ \oplus \\ \mathfrak{L} \end{matrix}.$$

The last condition is equivalent to

$$(U-T)A = XZ \quad (\text{and } TA = AZ). \tag{3.4'}$$

Since the space $\mathfrak{L}$ can be identified with $(D_T\mathfrak{H})^-$ and then the operator corresponding to $U-T$ is D_T, (3.4′) becomes

$$D_T A = XZ; \tag{3.4''}$$

here X is an operator from $\mathfrak{G}$ into $(D_T\mathfrak{H})^-$ (namely, the operator corresponding to the "original operator X"). Conditions (3.3) and (3.4″) are equivalent to the existence of a contraction $C: (D_A\mathfrak{G})^- \to (D_T\mathfrak{H})^-$ satisfying

$$X = CD_A, \tag{3.5}$$

$$D_T A = CD_A Z. \tag{3.6}$$

Since $\|D_T Ag\|^2 \leqq \|D_A Zg\|^2$ for all $g \in \mathfrak{G}$, it results that there exists a contraction defined on $(D_A Z\mathfrak{G})^-$ such that (3.6) holds. Obviously, this can be extended to a contraction $C: (D_A\mathfrak{G})^- \to (D_T\mathfrak{H})^-$. Then, if we define by (3.5) an operator $X: \mathfrak{G} \to (D_T\mathfrak{H})^-$, it is clear that $B_1 = \begin{bmatrix} A \\ X \end{bmatrix} \in \mathscr{B}_{T_{(1)}}(A)$.

By recurrence, we define, for every $n \geqq 1$,

$$\mathscr{B}_{T_{(n)}}(B_{n-1}) = \{B_n \in L(\mathfrak{G}, \mathfrak{H}_{(n)}) : T_{(n)} B_n = B_n Z,\ \|B_n\| = 1,\ P_{\mathfrak{H}_{(n-1)}} B_n = B_{n-1}\}, \tag{3.7}$$

where $B_0 = A$.

Remark 3.1. It is easy to show that if $B_n \in \mathscr{B}_{T_{(n)}}(B_{n-1})$ $(n=1, 2, \ldots)$ and if all B_n's are considered in $L(\mathfrak{G}, \mathfrak{K})$, then the strong limit $B = \lim_{n\to\infty} B_n$ exists; obviously, B is a dilation of A with $\|B\| = 1$. Also, since U is the strong limit of $(T_{(n)} P_{(n)})_{n=1}^{\infty}$, we clearly have $B \in I(U; Z)$. Thus, B defined as the strong limit of $(B_n)_{n=1}^{\infty}$, where $B_n \in \mathscr{B}_{T_{(n)}}(B_{n-1})$ $(n=1, 2, \ldots)$, is an EID of A. Conversely, for any EID B of A, the compression $B_n = P_{(n)} B$ belongs to $\mathscr{B}_{T_{(n)}}(B_{n-1})$ and B is the strong limit of $(B_n)_{n=1}^{\infty}$. [1])

Remark 3.2. It is plain that by the canonical identifications we have $(T_{(n)})_{(1)} = T_{(n+1)}$ and that for any $B_n \in \mathscr{B}_{T_{(n)}}(B_{n-1})$

$$\mathscr{B}_{T_{(n+1)}}(B_n) = \mathscr{B}_{(T_{(n)})_{(1)}}(B_n)$$

(for all $n = 1, 2, \ldots$).

Using the above remarks we shall obtain

Lemma 3.1. *A sufficient condition in order that* $A \in I(T; Z)$, $\|A\| = 1$, *have a unique EID is*

$$(D_A Z\mathfrak{G})^- = (D_A\mathfrak{G})^-. \tag{3.8}$$

Proof. We shall show by induction that, by virtue of (3.8), $B_n \in \mathscr{B}_{T_{(n)}}(B_{n-1})$ (where $\mathscr{B}_{T_{(n)}}(B_{n-1})$ is defined by (3.7)) is uniquely determined by A for every $n \geqq 1$. First, it is obvious by the construction of $B_1 = \begin{pmatrix} A \\ X \end{pmatrix} \in \mathscr{B}_{T_{(1)}}(A)$, where X is

[1]) This iterative explication of the construction of an EID, firstly given in [8], was inspired by [4].

defined by (3.5), that the contraction C of this formula is uniquely defined on $(D_A Z\mathfrak{G})^-$ by (3.6); therefore if (3.8) holds, then C is uniquely determined on the whole $(D_A\mathfrak{G})^-$. Consequently X, and thus B_1, is uniquely determined by $A=B_0$. From here, by the construction of $B_n \in \mathscr{B}_{T_{(n)}}(B_{n-1})$ $(n=1, 2, \ldots)$ and by virtue of Remark 3.2, we infer the following sufficient condition that B_n should be uniquely determined by its preceding B_{n-1}:

$$(D_{B_{n-1}}Z\mathfrak{G})^- = (D_{B_{n-1}}\mathfrak{G})^-. \tag{3.9}$$

Also we notice that

$$\begin{aligned}\|D_{B_n}(g-Zg')\|^2 &= \|g-Zg'\|^2 - \|B_n(g-Zg')\|^2 \leqq \\ &\leqq \|g-Zg'\|^2 - \|P_{\mathfrak{H}_{(n-1)}}B_n(g-Zg')\|^2 = \|D_{B_{n-1}}(g-Zg')\|^2 \leqq \ldots \\ &\ldots \leqq \|D_{B_1}(g-Zg')\|^2 \leqq \|D_A(g-Zg')\|^2,\end{aligned}$$

for all $g, g' \in \mathfrak{G}$ $(n=1, 2, \ldots)$. Hence, if (3.8) holds, (3.9) holds too, for all $n=1, 2, \ldots$. Now, let us assume that B_{n-1} is uniquely determined by A. Then, since by the above remark B_n is uniquely determined by B_{n-1}, it readily follows by our induction hypothesis that it is uniquely determined by A. From this and by virtue of Remark 3.1 we infer that A has a unique EID.

Now, returning to the original situation we can easily prove that the regularity condition imposed on one of the factorizations $A \cdot T_2$ or $T_1 \cdot A$ implies the uniqueness of the EID of A. First, let us assume that the factorization $A \cdot T_2$ of AT_2 is regular. Then, by Lemma 2.2, the factorization $\tilde{A} \cdot U_2$ of $\tilde{A}U_2$ is regular, and then, by Lemma 3.1, $\tilde{A}$ has a unique EID. Thus, by Remark 2.1, A also has a unique EID. Now, assume that the factorization $T_1 \cdot A$ of $T_1 A$ is regular. Then, it is known ([9], Ch. VII, § 2) that the factorization $A^* \cdot T_1^*$ is regular, and thus, by the same rasons as above, A^* has a unique EID. Consequently, by virtue of Lemma 2.1, so has A.

4. For the remaining part of Theorem 1.1, we have only to prove that if none of the factorizations $T_1 \cdot A$ and $A \cdot T_2$ (of $T_1 A = AT_2$) is regular, then the contraction A has at least two different EID's.

By virtue of Lemma 2.2 and Remark 2.3, our present assumption concerning the factorizations $T_1 \cdot A$ and $A \cdot T_2$ implies that the factorizations $T_1 \cdot \tilde{A}$ and $\tilde{A} \cdot U_2$, where $\tilde{A} = AP_{\mathfrak{H}_2} \in I(T_1; U_2)$ are not regular either. Also, by virtue of Remarks 2.1 and 3.1, it suffices to show that if the above conditions hold then $\mathscr{B}_{T_{(1)}}(\tilde{A})$ (defined by (3.1)) is not a singleton. We must show, by virtue of (3.2), (3.5), and (3.6), that the contraction C defined by

$$CD_{\tilde{A}}U_2 = D_{T_1}\tilde{A} \tag{4.1}$$

has at least one contractive extension $C' : (D_{\tilde{A}}\mathfrak{K}_2)^- \to (D_{T_1}\mathfrak{H}_1)^-$ such that

$$C' | (D_{\tilde{A}}\mathfrak{K}_2)^- \ominus (D_{\tilde{A}}U_2\mathfrak{K}_2)^- \neq 0. \tag{4.2}$$

Since the factorization $T_1 \cdot \tilde{A}$ does not satisfy (2.9), there exist $h_0 \in (D_{T_1}\mathfrak{H}_1)^-$ and $k_0 \in \mathfrak{H}_2$ such that

(4.3) $$D_{T_1} h_0 = D_{\tilde{A}^*} k_0 \neq 0;$$

also, since the factorization $\tilde{A} \cdot U_2$ does not satisfy (1.2), there exists $0 \neq d_0 \in (D_{\tilde{A}}\mathfrak{K}_2)^- \ominus \ominus (D_{\tilde{A}} U_2 \mathfrak{K}_2)^-$, where we can suppose that $\|h_0\|=1$ and $\|d_0\|=1$. Now, we define $C': (D_{\tilde{A}}\mathfrak{K}_2)^- \to (D_{T_1}\mathfrak{H}_1)^-$ by

(4.4) $$C' = CQ + \theta d_0^* \otimes h_0$$

where Q is the orthogonal projection of $(D_{\tilde{A}}\mathfrak{K}_2)^-$ onto $(D_{\tilde{A}} U_2 \mathfrak{K}_2)^-$, $d_0^* \otimes h_0$ is the operator defined on $(D_{\tilde{A}}\mathfrak{K}_2)^-$ by $(d_0^* \otimes h_0)d = (d, d_0)h_0$, and $0 < \theta < 1$ will be chosen later. Obviously, $C'd_0 \neq 0$, thus (4.2) holds. Also, we shall show that θ can be chosen such that C' defined by (4.4) be a contraction, i.e.

$$\|CQd + \theta((I-Q)d, d_0)h_0\| \leqq \|d\|,$$

or equivalently,

(4.5) $$\|CQd\|^2 + 2\theta \operatorname{Re} (CQd, h_0)\, \overline{((I-Q)d, d_0)} + \theta^2 |((I-Q)d, d_0)|^2 \leqq$$
$$\leqq \|Qd\|^2 + \|(I-Q)d\|^2, \quad \text{for all} \quad d \in (D_{\tilde{A}}\mathfrak{K}_2)^-.$$

Obviously, it is enough to verify (4.5) for d of the form $D_{\tilde{A}} U_2 k + \lambda d_0$ ($k \in \mathfrak{K}_2$, $\lambda \in \mathbf{C}$), for which (4.5) becomes

$$\|CD_{\tilde{A}} U_2 k\|^2 + 2\theta \operatorname{Re} \bar{\lambda} (CD_{\tilde{A}} U_2 k, h_0) + \theta^2 |\lambda|^2 \leqq \|D_{\tilde{A}} U_2 k\|^2 + |\lambda|^2,$$

or according to (4.1),

(4.6) $$2\theta \operatorname{Re} \bar{\lambda} (D_{T_1} \tilde{A} k, h_0) \leqq \|D_{\tilde{A}} U_2 k\|^2 - \|D_{T_1} \tilde{A} k\|^2 + |\lambda|^2 (1-\theta^2) =$$
$$= \|D_{\tilde{A}} k\|^2 + |\lambda|^2 (1-\theta^2) \quad (k \in \mathfrak{K}_2, \lambda \in \mathbf{C}).$$

It is elementary to deduce that (4.6) is true if

(4.7) $$|(D_{T_1} \tilde{A} k, h_0)|^2 \leqq \|D_{\tilde{A}} k\|^2 (1-\theta^2)\theta^{-2} \quad (k \in \mathfrak{K}_2).$$

Since by (4.3) we have $(D_{T_1} \tilde{A} k, h_0) = (D_{\tilde{A}} k, \tilde{A}^* k_0)$ for all $k \in \mathfrak{K}_2$, it is easy to prove that (4.7) will be true if we choose $0 < \theta < (1 + \|\tilde{A}^* k_0\|^2)^{-1/2}$. This concludes the proof of Theorem 1.1.

Remark 4.1. Plainly, the whole proof in this section works for any contraction $A \in I(T_1; T_2)$. Also, if for such an A, one of the factorizations $A \cdot T_2$ and $T_1 \cdot A$ of $T_1 A = A T_2$ is regular then either $\|A\|=1$ or T_2 is a coisometry or T_1 is an isometry. By virtue of Theorem 1.1 and Lemma 2.1 we infer that in any of these cases A has exactly one contractive intertwining dilation $\in I(U_1; U_2)$. Thus, we can reformulate Theorem 1.1 in the following, slightly more general form: *A contraction $A \in I(T_1; T_2)$ has a unique contractive intertwining dilation $\in I(U_1; U_2)$ if and only if at least one of the factorizations $T_1 \cdot A$ and $A \cdot T_2$ of $T_1 A = A T_2$ is regular.*

Remark 4.2. We give an example showing that it is not necessary that both factorizations $A\cdot T_2$ and $T_1\cdot A$ be regular in order to have the uniqueness property of the EID of A.

To this purpose we define $A\in L(l^2)$, by

$$A(c_0, c_1, \dots, c_n, \dots) = (c_0, (1-d_1^2)^{1/2}c_1, \dots (1-d_n^2)^{1/2}c_n, \dots)$$

where $x=(c_n)_{n=0}^{\infty}\in l^2$ and $0<d_n<d_{n+1}<1$ $(n=1, 2, \dots)$ are fixed. Also we denote by $T\in L(l^2)$ the weighted shift

$$T(c_0, c_1, \dots, c_n, \dots) = (0, (1-d_1^2)^{1/2}c_0, \dots, (1-d_n^2)^{1/2}(1-d_{n-1}^2)^{-1/2}c_{n-1}, \dots)$$

and by U the unilateral shift

$$U(c_0, c_1, \dots, c_n, \dots) = (0, c_0, \dots, c_{n-1}, \dots)$$

on l^2. Then, clearly, A and T are contractions on l^2 and U is an isometry. Also, it is easy to verify that $TA=AU, A^*=A$, $\|A\|=1$ and

$$T^*(c_0, c_1, \dots, c_n, \dots) = ((1-d_1^2)^{1/2}c_1, \dots, (1-d_{n+1}^2)^{1/2}(1-d_n^2)^{-1/2}c_{n+1}, \dots).$$

Then, we obtain

$$D_A(c_0, c_1, \dots, c_n, \dots) = (0, d_1c_1, \dots, d_nc_n, \dots),$$
$$D_T(c_0, c_1, \dots, c_n, \dots) =$$
$$= (d_1c_0, (d_2^2-d_1^2)^{1/2}(1-d_1^2)^{-1/2}c_1, \dots, (d_{n+1}^2-d_n^2)^{1/2}(1-d_n^2)^{-1/2}c_n, \dots).$$

Whence, obviously

$$D_A l^2\cap D_{U^*}l^2 = D_A l^2\cap \ker U^* = \{0\}, \tag{4.8}$$

$$D_T l^2\cap D_{A^*}l^2 \ni (0, 1, 0, \dots). \tag{4.9}$$

Therefore, by virtue of Remark 2.2, we infer from (4.8), respectively from (4.9), that the factorization $A\cdot U$, respectively $T\cdot A$, (of $AU=TA$) is regular, respectively nonregular.

5. Let us notice that Theorem 1.1 has the following direct consequences:

Corollary 5.1. *Let A and T be double commuting* (i.e. $AT=TA$, $AT^*=T^*A$) *contractions on $\mathfrak{H}$, $\|A\|=1$. Then A has a unique exact intertwining dilation (with respect to $T_1=T=T_2$) if and only if there is a decomposition $\mathfrak{H}=\mathfrak{H}_A\oplus\mathfrak{H}_T$ reducing A and T, such that $A|\mathfrak{H}_A$ and $T^*|\mathfrak{H}_T$ are isometric or that $A^*|\mathfrak{H}_A$ and $T|\mathfrak{H}_T$ are isometric.*

Indeed, the splitting properties obviously imply

$$D_AD_{T^*} = D_{T^*}D_A = 0, \tag{5.1}$$

respectively

$$D_TD_{A^*} = D_{A^*}D_T = 0. \tag{5.2}$$

Conversely, if (5.1), respectively (5.2), is satisfied, then defining $\mathfrak{H}_A$ as the smallest (linear closed) subspace of $\mathfrak{H}$ reducing T and containing $D_{T^*}\mathfrak{H}$, respectively reducing A and containing $D_{A^*}\mathfrak{H}$, we obtain the splitting properties stated above.

By the double commuting property, (5.1), respectively (5.2), is equivalent to

$$D_A\mathfrak{H}\cap D_{T^*}\mathfrak{H} = \{0\}, \quad \text{respectively} \quad D_T\mathfrak{H}\cap D_{A^*}\mathfrak{H} = \{0\},$$

thus, by Remark 2.2, to the regularity of the factorization $A\cdot T$, respectively $T\cdot A$, of $AT=TA$.

Corollary 5.2. *Let A, $T\in L(\mathfrak{H})$ be commuting contractions. Then A has a unique contractive intertwining dilation (with respect to T) if and only if T has a unique contractive intertwining dilation (with respect to A).*

Indeed, by Remark 4.1 each of the two assertions above is equivalent to the regularity of at least one of the factorizations $A\cdot T$ or $T\cdot A$ of $AT=TA$.

Corollary 5.3. *Let $A\in L(\mathfrak{H}_2, \mathfrak{H}_1)$, $\|A\|=1$, intertwine the coisometry T_1 and the isometry T_2. Then A has a unique exact intertwining dilation if and only if at least one of the following two conditions holds:*

$$D_A\mathfrak{H}_2\cap \ker T_2^* = \{0\}, \quad D_{A^*}\mathfrak{H}_1\cap \ker T_1 = \{0\}.$$

Indeed, under the present assumptions, these conditions are equivalent to the regularity of the factorizations $A\cdot T_2$, respectively $T_1\cdot A$ of $AT_2=T_1A$ (see Remark 2.2).

Remark 5.1. The preceding corollary is a slight extension of the uniqueness theorem of ADAMJAN, AROV and KREIN, [2] Theorem 3.1, which concerns the case when T_2 and T_1^* are unilateral shifts. However, in case $T_2\in C_{\cdot 0}$, $T_1\in C_{0\cdot}$ (i.e. if $T_2^{*n}\to 0$, $T_1^n\to 0$ strongly, for $n\to\infty$) our Theorem 1.1 is an easy consequence of [2], Theorem 3.1 and [9], Ch. II, Theorem 1.2.

Let us also indicate how one of the main results of [3] follows from our Theorem 1.1. To this purpose we recall that according to [3], a contraction $A\in L(\mathfrak{H}_2, \mathfrak{H}_1)$ is said to *Harnack-dominate* a contraction $B\in L(\mathfrak{H}_2, \mathfrak{H}_1)$ if there exists a positive constant γ such that

$$\|D_Bh\| \leqq \gamma\|D_Ah\| \quad \text{and} \quad \|(B-A)h\| \leqq \gamma\|D_Ah\| \quad (h\in\mathfrak{H}_2). \tag{5.3}$$

Plainly, relations (5.3) imply that

$$D_B\mathfrak{H}_2\subset D_A\mathfrak{H}_2 \quad \text{and} \quad (B-A)^*\mathfrak{H}_1\subset D_A\mathfrak{H}_2. \tag{5.4}$$

Corollary 5.4. ([3], Theorem 3.2) *Let A, $B\in L(\mathfrak{H}_2, \mathfrak{H}_1)$ intertwine the contractions T_1 and T_2, $\|A\|=1$, and such that A Harnack-dominates B. Then if A has a unique EID so has B.*

Proof. By Theorem 1.1, one of the factorizations $A \cdot T_2$ and $T_1 \cdot A$ is regular. If the first one is regular, then from (2.9) (with $A_2=A, A_1=T$ and $A_2=B, A_1=T$) and from the first relation (5.4) we readily infer that the factorization $B \cdot T_2$ is regular, thus by Theorem 1.1, B has a unique EID. In case $T_1 \cdot A$ is regular, from (2.10) (with $A_2=T_1, A_1=A$) we obtain

$$D_{T_1}\mathfrak{H}_1 \cap \ker A^* = \{0\}, \quad D_A\mathfrak{H}_2 \cap A^* D_{T_1}\mathfrak{H}_1 = \{0\}. \tag{5.5}$$

If

$$B^* D_{T_1} h_1 = 0 \quad \text{and} \quad D_B h_2 = B^* D_{T_1} h_1'$$

for some $h_1, h_1' \in \mathfrak{H}_1, h_2 \in \mathfrak{H}_2$, then from (5.4) we infer at once that

$$A^* D_{T_1} h_1 \in D_A \mathfrak{H}_2 \quad \text{and} \quad A^* D_{T_1} h_1' \in D_A \mathfrak{H}_2;$$

by (5.5), it follows $D_{T_1} h_1 = 0 = D_{T_1} h_1'$. We conclude that $A_2=T_1, A_1=B$ satisfy (2.10), thus that the factorization $T_1 \cdot B$ is regular. Since (5.3) also implies $\|B\|=1$, the proof is achieved by referring to Theorem 1.1.

6. A less direct consequence of our preceding results is the following

Proposition 6.1. *Let $A \in L(\mathfrak{H}_2, \mathfrak{H}_1)$, $\|A\|=1$, intertwine the contractions $T_1 \in L(\mathfrak{H}_1)$ and $T_2 \in L(\mathfrak{H}_2)$ and let $\mathfrak{M}$ be a subspace of $\mathfrak{H}_2$, cyclic for the minimal unitary dilation U_2 of T_2. If, moreover, $\mathfrak{M}$ enjoys also the property*

$$D_A\mathfrak{M} \oplus \{0\} \subset \{D_A T_2 h \oplus D_{T_2} h : h \in \mathfrak{M}\}^-, \tag{6.1}$$

then A has a unique exact intertwining dilation.

Proof. We shall use the notations of the preceding sections. In particular we set $\tilde{A} = AP_{\mathfrak{H}_2}$. Also we set

$$\mathfrak{K}_2' = \bigvee_{n=0}^{\infty} U_2^n \mathfrak{M} \tag{6.2}$$

and

$$U_2' = U_2|\mathfrak{K}_2', \quad \tilde{A}' = \tilde{A}|\mathfrak{K}_2'.$$

For elements $h \in \mathfrak{M}$ and $k \in \mathfrak{K}_2'$ of the form

$$k = \sum_{n=0}^{\infty} U_2^n k_n, \tag{6.3}$$

where $k_n \in \mathfrak{M}$ $(n=0, 1, 2, \ldots)$ and only a finite number of k_n's are $\neq 0$, we have

$$\begin{aligned} \|D_{\tilde{A}'}[k - U_2'(k_1 + h + \sum_{n=2}^{\infty} U_2^{n-1} k_n)]\|^2 = \\ = \|D_{\tilde{A}}(k - \sum_{n=1}^{\infty} U_2^n k_n - U_2 h)\|^2 = \|D_{\tilde{A}}(k_0 - U_2 h)\|^2 = \\ = \|k_0 - T_2 h\|^2 + \|(U_2 - T_2) h\|^2 - \|A(k_0 - T_2 h)\|^2 = \\ = \|D_A(k_0 - T_2 h)\|^2 + \|D_{T_2} h\|^2 = \|D_A k_0 \oplus 0 - D_A T_2 h \oplus D_{T_2} h\|^2. \end{aligned} \tag{6.4}$$

The last quantity can be made, by virtue of (6.1), as small as we want if $h\in\mathfrak{M}$ is suitably chosen. Thus, we can deduce from (6.4) that the factorization $\tilde{A}'\cdot U_2'$ is regular. Consequently, from Theorem 1.1 it follows that $\tilde{A}'$ has a unique EID; let B' be this EID. It enjoys the property

$$(6.5)\qquad P_{\mathfrak{H}_1}B'=\tilde{A}' \quad\text{and}\quad U_1B'=B'U_2'.$$

Let now B_j $(j=1,2)$ be two EID of A. As we already pointed out in Section 2, there exists a unique contractive extension $\hat{B}_j\in L(\hat{\mathfrak{K}}_2,\hat{\mathfrak{K}}_1)$ such that

$$(6.6)\qquad \|\hat{B}_j\|=\|B_j\|,\quad \hat{B}_j\hat{U}_2=\hat{U}_1\hat{B}_j\quad (j=1,2).$$

Since $\hat{B}_j|\mathfrak{K}_2'$ is a contraction from $\mathfrak{K}_2'$ into $\mathfrak{K}_1$ enjoying property (6.5), by the uniqueness of B' we infer

$$(6.7)\qquad \hat{B}_1|\mathfrak{K}_2'=B_1|\mathfrak{K}_2'=B'=B_2|\mathfrak{K}_2'=\hat{B}_2|\mathfrak{K}_2';$$

whence, by (6.6),

$$(6.8)\qquad \hat{B}_1g=\hat{B}_2g$$

for any element $g\in\hat{\mathfrak{K}}_2$ of the form

$$(6.9)\qquad g=\hat{U}_2^nk'\quad (\text{with } n=0,\pm1,\pm2,\ldots;\ k'\in\mathfrak{K}_2').$$

Since $\mathfrak{K}_2'$ contains $\mathfrak{M}$ which is cyclic for U_2, the elements g of the form (6.8) span $\hat{\mathfrak{K}}_2$, thus from (6.6) and (6.8) we deduce that $\hat{B}_1=\hat{B}_2$, and hence $B_1=B_2$. This shows that A has a unique EID and thus the proof is achieved.

Remark 6.1. In case $\mathfrak{M}$ is an invariant subspace for T_2, then (6.1) is equivalent to the regularity of the factorization $(A|\mathfrak{M})\cdot(T_2|\mathfrak{M})$ of $AT_2|\mathfrak{M}$.

Corollary 6.1. *Let A be a contraction intertwining the contractions T_1 and T_2. Then, if* $\ker D_A$ *is cyclic for the unitary dilation $\hat{U}_2$ of T_2, A has a unique exact intertwining dilation.*

Indeed, in this case, for $\mathfrak{M}=\ker D_A$, the left hand side of (6.1) is $\{0\}\oplus\{0\}$ and consequently (6.1) is trivially satisfied.

Remark 6.2. Corollary 6.1 (which however can be easily proved in a direct way by an argument similar to the last part of the proof of Proposition 6.1) contains as particular cases some uniqueness theorems of [1] and [5].

References

[1] V. M. Adamjan—D. Z. Arov—M. G. Krein, Infinite Hankel matrices and generalized Carathéodory—Fejér and Schur problems, *Funkc. Anal. Priložen.*, 2:4 (1968), 1—17. (Russian)

[2] V. M. Adamjan—D. Z. Arov—M. G. Krein, Infinite Hankel block-matrices and related continuation problems, *Izv. Akad. Nauk Armjan. SSR, Matematika*, **6** (1971), 87—112. (Russian)

[3] Z. Ceauşescu, On intertwining dilations, *Acta Sci. Math.*, **38** (1976), 281—290.

[4] R. G. Douglas—P. S. Muhly—C. Pearcy, Lifting commuting operators, *Michigan Math. J.*, **15** (1968), 385—395.

[5] D. Sarason, Generalized interpolation in H^∞, *Trans. Amer. Math. Soc.*, **127** (1967), 179—203.

[6] Ja. S. Švarcman, On invariant subspaces of a dissipative operator and the divisors of its characteristic function, *Funkc. Anal. i Priložen.*, 4:4 (1970), 85—86. (Russian)

[7] B. Sz.-Nagy—C. Foiaş, Forme triangulaire d'une contraction et factorisation de la fonction caractéristique, *Acta Sci. Math.*, **28** (1967), 201—212.

[8] B. Sz.-Nagy—C. Foiaş, Dilatation des commutants d'opérateurs, *C. R. Acad. Sci. Paris*, Série A, **266** (1968), 493—495.

[9] B. Sz.-Nagy—C. Foiaş, *Harmonic analysis of operators on Hilbert space* (Amsterdam—Budapest, 1970).

[10] B. Sz.-Nagy—C. Foiaş, Regular factorizations of contractions, *Proc. Amer. Math. Soc.*, **43** (1974), 91—93.

T. ANDO
INST. APPLIED ELECTRICITY
HOKKAIDO UNIVERSITY
SAPPORO, JAPAN

Z. CEAUŞESCU
„INCREST"
CALEA VICTORIEI 114
BUCHAREST, ROMANIA

C. FOIAŞ
DEPARTMENT OF MATH.
UNIVERSITY BUCHAREST
STR. ACADEMIEI 14
BUCHAREST, ROMANIA