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Star-algebras induced by non-degenerate inner products

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§ 1. Introduction

Let $\mathscr{B}(\mathfrak{E})$ denote the algebra of all bounded linear operators on the Banach space $\mathfrak{E}$. According to a classical theorem of Kawada, Kakutani and Mackey (see e.g. [1; Corollary 4.10.8]), if * is an involution on $\mathscr{B}(\mathfrak{E})$ satisfying the condition $T^*T\neq 0$ for every non-zero $T\in\mathscr{B}(\mathfrak{E})$, then there is a positive definite inner product $(\cdot,\cdot)$ on $\mathfrak{E}$ such that

(i) $(Tx, y)=(x, T^*y)$ for every $x, y\in\mathfrak{E}$ and $T\in\mathscr{B}(\mathfrak{E})$;

(ii) the norm induced by $(\cdot,\cdot)$ is equivalent to the original norm on $\mathfrak{E}$.

In our paper [2] we generalized this theorem to a class of indefinite inner products and began similar investigations for wider classes.

J. SARANEN [3] has obtained numerous further improvements and generalizations involving non-symmetric bilinear forms as well as operator algebras different from $\mathscr{B}(\mathfrak{E})$ on normed or non-normed vector spaces.

Below, combining the stand-point and methods of [2] with achievements of [3], we try to give a unified, elementary and possibly complete treatment of those aspects of the subject which are relevant to the general theory of indefinite inner product spaces [4]. For this purpose, we single out certain results explicit or implicit in [3], regroup, reformulate, extend or restrict them, modify their proofs, and add some new observations (cf. especially Theorems 3.5, 3.7, 4.7, 4.10 and some corollaries).

It should be noted that representations for involutions of general (i.e., not operator) algebras by means of indefinite inner products have been known prior to [2] (see [1; Theorem 4.3.7]). However, they seem to be of a different nature, since their representation space is not fixed in advance.

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§ 2. Preliminaries

1. Admissible *-algebras. Let $\mathfrak{E}$ be a vector space over the complex field $\mathbf{C}$. The algebra of all linear operators (i.e., all homomorphisms) $T: \mathfrak{E} \to \mathfrak{E}$ will be denoted by $\mathscr{L}(\mathfrak{E})$.

Let $\mathscr{A}$ be a subalgebra of $\mathscr{L}(\mathfrak{E})$. The mapping *: $\mathscr{A} \to \mathscr{A}$ is said to be an *involution* if for all T_1, T_2, $T \in \mathscr{A}$ and $\alpha \in \mathbf{C}$ the following conditions are satisfied: (i) $(T_1+T_2)^* = T_1^* + T_2^*$, (ii) $(\alpha T)^* = \bar{\alpha} T^*$, (iii) $(T_1 T_2)^* = T_2^* T_1^*$, (iv) $T^{**} = T$.

An algebra equipped with an involution * is called a **-algebra.*

The algebra (or *-algebra) $\mathscr{A} \subset \mathscr{L}(\mathfrak{E})$ is said to be *dense* if, for any positive integer n, linearly independent vectors $x_1, \dots, x_n \in \mathfrak{E}$ and vectors $y_1, \dots, y_n \in \mathfrak{E}$, there is an operator $T \in \mathscr{A}$ such that $Tx_j = y_j$ $(j=1, \dots, n)$.

$\mathscr{L}(\mathfrak{E})$ itself is a dense algebra. What is more, the finite-rank elements of $\mathscr{L}(\mathfrak{E})$ also form a dense algebra (see [3; Lemma 2.2]). If $\mathfrak{E}$ is a Banach space, the algebra $\mathscr{B}(\mathfrak{E})$ of all *bounded* linear operators $T: \mathfrak{E} \to \mathfrak{E}$ is dense. Even the finite-rank elements of $\mathscr{B}(\mathfrak{E})$ form a dense algebra ([3; Lemma 2.2]).

We say the algebra (*-algebra) $\mathscr{A} \subset \mathscr{L}(\mathfrak{E})$ is *admissible* if $\mathscr{A}$ is dense and contains an operator of rank 1.

2. Non-degenerate inner products. Let $\mathfrak{E}$ be a vector space over $\mathbf{C}$. We say a mapping of $\mathfrak{E} \times \mathfrak{E}$ into $\mathbf{C}$ is an *inner product* if, denoting the image of the ordered pair $x, y \in \mathfrak{E}$ by (x, y), for any $x_1, x_2, x, y \in \mathfrak{E}$ and $\alpha \in \mathbf{C}$ the following conditions are fulfilled: (i) $(x_1+x_2, y) = (x_1, y) + (x_2, y)$, (ii) $(\alpha x, y) = \alpha(x, y)$, (iii) $(y, x) = \overline{(x, y)}$.

The inner product $(\cdot, \cdot)$ is said to be *non-degenerate* if for $x \neq 0$ there exists $y \in \mathfrak{E}$ such that $(x, y) \neq 0$.

A norm $|\cdot|$ is said to be *compatible* with the non-degenerate inner product $(\cdot, \cdot)$ on $\mathfrak{E}$ if (i) for any fixed $y \in \mathfrak{E}$ the linear form $\varphi_y(x) = (x, y)$ $(x \in \mathfrak{E})$ is continuous in the norm $|\cdot|$, (ii) for any $|\cdot|$-continuous linear form φ there exists $y \in \mathfrak{E}$ satisfying the relation $\varphi = \varphi_y$.

Here we note that if (x, y) is $|\cdot|$-continuous in the variable x then it is $|\cdot|$-continuous in y as well (since $(y, x) = \overline{(x, y)}$) and, in case $\mathfrak{E}$ is complete for $|\cdot|$, it is jointly $|\cdot|$-continuous in x and y (a consequence of the principle of uniform boundedness; see [4; Theorem IV. 2.3]).

3. Induced *-algebras. Let $(\cdot, \cdot)$ be a non-degenerate inner product on the vector space $\mathfrak{E}$. Given a linear operator $T \in \mathscr{L}(\mathfrak{E})$ it may happen that for each $y \in \mathfrak{E}$ there is a $y_T \in \mathfrak{E}$ with the property

$$(Tx, y) = (x, y_T) \quad (x \in \mathfrak{E}). \tag{2.1}$$

By the non-degeneracy of the inner product, the vector y_T is unique. The relation

$T^{()}y=y_T$ $(y\in\mathfrak{E})$ defines a linear operator $T^{()}\in\mathscr{L}(\mathfrak{E})$. Thus the existence, for each $y\in\mathfrak{E}$, of a vector $y_T\in\mathfrak{E}$ with property (2.1) is equivalent to the existence of a linear operator $T^{()}\in\mathscr{L}(\mathfrak{E})$ satisfying the condition

(2.2) $$(Tx, y) = (x, T^{()}y) \quad (x, y\in\mathfrak{E}).$$

Obviously, $T^{()}$ is unique; it is called the *adjoint* of T relative to the inner product $(\cdot, \cdot)$.

We write $\mathrm{Ind}_{()}$ for the set of all operators $T\in\mathscr{L}(\mathfrak{E})$ which do have an adjoint relative to $(\cdot, \cdot)$:

(2.3) $$\mathrm{Ind}_{()} = \{T\in\mathscr{L}(\mathfrak{E}) : T^{()} \text{ exists}\}.$$

It is easy to see that $\mathrm{Ind}_{()}$ is an algebra and that the mapping $T\mapsto T^{()}$ $(T\in\mathrm{Ind}_{()})$ is an involution on $\mathrm{Ind}_{()}$. We say $\mathrm{Ind}_{()}$ is the *-algebra *induced* by the non-degenerate inner product $(\cdot, \cdot)$.

4. Inner products representing a *-algebra. Let $\mathscr{A}\subset\mathscr{L}(\mathfrak{E})$ be a *-algebra. If there exists a non-degenerate inner product $(\cdot, \cdot)$ on $\mathfrak{E}$ satisfying

(2.4) $$(Tx, y) = (x, T^*y) \quad (T\in\mathscr{A};\ x, y\in\mathfrak{E}),$$

we say $(.,.)$ *represents* the *-algebra $\mathscr{A}$ (or the involution *).

In other words, $(\cdot, \cdot)$ represents $\mathscr{A}$ if $\mathscr{A}$ is a *-subalgebra of $\mathrm{Ind}_{()}$. In this case we write $\mathscr{A}\subset\mathrm{Ind}_{()}$. (More generally, if $\mathscr{A}_1$ and $\mathscr{A}_2$ are *-algebras, then $\mathscr{A}_1\subset\mathscr{A}_2$ will signify that $\mathscr{A}_1$ is a *-subalgebra of $\mathscr{A}_2$.)

5. Decomposable inner products. Let $(\cdot, \cdot)$ be a non-degenerate inner product on the vector space $\mathfrak{E}$.

Two vectors $x, y\in\mathfrak{E}$ are said to be *orthogonal* if $(x, y)=0$. Two subsets $\mathfrak{A}$, $\mathfrak{B}\subset\mathfrak{E}$ are said to be orthogonal if $(x, y)=0$ for all $x\in\mathfrak{A}$ and $y\in\mathfrak{B}$.

We say that the (linear) subspace $\mathfrak{M}\subset\mathfrak{E}$ is *positive definite* (or that $(\cdot, \cdot)$ is positive definite on $\mathfrak{M}$) if $(x, x)>0$ for all $x\in\mathfrak{M}$, $x\neq 0$. The definition of *negative definite* subspace is similar. A subspace is said to be *definite* if it is either positive definite or negative definite.

The subspace $\mathfrak{M}\subset\mathfrak{E}$ is said to be *neutral* if $(x, x)=0$ for all $x\in\mathfrak{M}$.

In case $\mathfrak{E}$ is the orthogonal direct sum of a positive definite subspace $\mathfrak{E}^+$ and a negative definite subspace $\mathfrak{E}^-$,

(2.5) $$\mathfrak{E} = \mathfrak{E}^+(\dot{+})\mathfrak{E}^-,$$

we say the space $\mathfrak{E}$ (or: the inner product $(\cdot, \cdot)$) is *decomposable* and (2.5) is a *fundamental decomposition.*

Let P^+ denote the projection to $\mathfrak{E}^+$ along $\mathfrak{E}^-$, and set $P^-=I-P^+$. Then the operator $J=P^+-P^-$ has the properties $J^2=I$, $(Jx, y)=(x, Jy)$ for all $x, y\in\mathfrak{E}$.

Moreover, $(Jx, x)>0$ if $x\neq 0$. The positive definite inner product

$$(x, y)_J = (Jx, y) \quad (x, y\in\mathfrak{E}) \tag{2.6}$$

and the norm

$$x)^{1/2}\|x\|_J = (Jx, \quad (x\in\mathfrak{E}) \tag{2.7}$$

will be called the *fundamental inner product* and the *fundamental norm* corresponding to (2.5).

If, for some fundamental norm, $\mathfrak{E}$ is complete, we say $\mathfrak{E}$ is a *Krein space.*

All fundamental norms on a Krein space are topologically equivalent (cf. [4; Corollary IV. 6.3]).

If $\mathfrak{E}$ is a decomposable space and $\mathfrak{E}^+$ or $\mathfrak{E}^-$ has finite dimension, the space (or: the inner product) is said to be *quasi-definite;* the non-negative integer

$$\varkappa(\mathfrak{E}) = \min\{\dim\mathfrak{E}^+, \dim\mathfrak{E}^-\} \tag{2.8}$$

is called the *rank of indefiniteness.*

Quasi-definiteness and the value (2.8) do not depend on the choice of the fundamental decomposition (2.5) (cf. [4; Corollary II.10.4]).

A quasi-definite Krein space is called a *Pontrjagin space.*

§ 3. Star-algebras on vector spaces

In this section, $\mathfrak{E}$ is a vector space over $\mathbf{C}$.

1. Admissible *-algebras in general. We first examine the problem of representing admissible *-algebras on $\mathfrak{E}$ without any additional assumption.

Theorem 3.1 (cf. [3; Folgerung 3.2 and relations (2.3a)—(2.3b)]). *Let* $(\cdot, \cdot)$ *be a non-degenerate inner product on* $\mathfrak{E}$. *Then* $\mathrm{Ind}_{()}$ *is an admissible* **-algebra on* $\mathfrak{E}$.

Proof. We mentioned in Section 2 that $\mathrm{Ind}_{()}$ is an algebra and $T\mapsto T^{()}$ $(T\in\mathrm{Ind}_{()})$ is an involution on $\mathrm{Ind}_{()}$.

Set

$$Rx = \sum_{j=1}^{r}(x, y_j)z_j \quad (x\in\mathfrak{E}), \tag{3.1}$$

where $y_j, z_j\in\mathfrak{E}$ $(j=1, \dots, r)$. Then $R\in\mathrm{Ind}_{()}$, since

$$(Rx, u) = \left(x, \sum_{j=1}^{r}(u, z_j)y_j\right) \quad (x, u\in\mathfrak{E}).$$

Moreover, in the case $r=1$ the operator R has rank 1. Finally, let us be given $2n$ vectors $x_k, w_k\in\mathfrak{E}$ $(k=1, \dots, n)$, the system $\{x_1, \dots, x_n\}$ being linearly independent.

Choose $y_1, \dots, y_n$ such that $(x_k, y_j) = \delta_{kj}$ $(j, k = 1, \dots, n$; see e.g. [4; Lemmas I.10.4 and I.10.6]). Then (3.1) with $r = n$ and $z_j = w_j$ $(j = 1, \dots, n)$ yields $Rx_k = w_k$ $(k = 1, \dots, n)$.

The following result is, in a certain sense, converse to Theorem 3.1.

Theorem 3.2 (cf. [3; Satz 5.1 and Satz 5.3]). *Let $\mathscr{A} \subset \mathscr{L}(\mathfrak{E})$ be an admissible *-algebra. Then there is one and, up to a constant real factor, only one non-degenerate inner product on $\mathfrak{E}$ which represents $\mathscr{A}$.*

Proof (cf. [2; pp. 56—60]). We first show that there is an operator T_0 with the properties

$$T_0 \in \mathscr{A}, \quad \dim T_0 \mathfrak{E} = 1, \quad T_0^* T_0 \neq 0. \tag{3.2}$$

By assumption, $\mathscr{A}$ contains an operator T_1 of rank 1. The operator T_1^* is non-zero, since $T_1^{**} = T_1$ is non-zero whereas $0^* = 0$. Choose vectors $e \notin \mathfrak{N}(T_1)$, $e_* \notin \mathfrak{N}(T_1^*)$, where $\mathfrak{N}(T)$ denotes the kernel of T. The algebra $\mathscr{A}$ being dense, there exists $Q \in \mathscr{A}$ such that $QT_1 e = e_*$. Set $T_2 = QT_1$.

As $\mathfrak{N}(T_2) \supset \mathfrak{N}(T_1)$, the operators T_2, $T_1 + T_2$, $T_1 + iT_2$ have rank not greater than 1. Therefore if (3.2) cannot be fulfilled then

$$T_1^* T_1 = T_2^* T_2 = (T_1 + T_2)^* (T_1 + T_2) = (T_1 + iT_2)^* (T_1 + iT_2) = 0.$$

Hence $T_1^* T_2 = 0$. On the other hand, the vector $T_1^* T_2 e = T_1^* Q T_1 e = T_1^* e_*$ is non-zero. Contradiction.

So let T_0 satisfy (3.2). Take

$$f \notin \mathfrak{N}(T_0^* T_0), \quad g = T_0^* T_0 f. \tag{3.3}$$

By assumption, for every $x \in \mathfrak{E}$ there exists $Q_x \in \mathscr{A}$ such that $Q_x g = x$. Set

$$P_x = Q_x T_0^* T_0. \tag{3.4}$$

Then

$$P_x \in \mathscr{A}, \quad P_x f = x, \quad P_x \mathfrak{N}(T_0) = 0 \quad (x \in \mathfrak{E}). \tag{3.5}$$

Relation (3.4) implies $P_x^* = T_0^* T_0 Q_x^*$. In particular, $P_x^* \mathfrak{E} \subset T_0^* T_0 \mathfrak{E} = \langle g \rangle$, the span of g. Thus

$$P_x^* y = \varphi_x(y) g \quad (x, y \in \mathfrak{E}), \tag{3.6}$$

where $\varphi_x : \mathfrak{E} \to \mathbf{C}$ is a linear form depending on x.

From (3.6), (3.5), (3.3) and (3.2) we obtain

$$P_x^* P_y = \varphi_x(y) T_0^* T_0 \quad (x, y \in \mathfrak{E}). \tag{3.7}$$

Really, the two sides of (3.7) coincide on f and $\mathfrak{N}(T_0)$ while the span of f and $\mathfrak{N}(T_0)$ equals $\mathfrak{E}$.

Suppose $(\cdot, \cdot)$ is a non-degenerate inner product representing $\mathscr{A}$. Then, in particular, $(P_x f, y)=(f, P_x^* y)$ for all x, y. Hence, on account of (3.5) and (3.6),

$$(x, y) = \overline{\varphi_x(y)}(f, g) \quad (x, y \in \mathfrak{E}), \tag{3.8}$$

where $(f, g)=(f, T_0^* T_0 f)=(T_0 f, T_0 f)$, a real number. This proves the uniqueness assertion.

To prove existence, choose a non-zero real number λ and set

$$(x, y) = \lambda \overline{\varphi_x(y)} \quad (x, y \in \mathfrak{E}). \tag{3.9}$$

From (3.5)—(3.6) it follows that $\varphi_{x_1+x_2}=\varphi_{x_1}+\varphi_{x_2}$ and $\varphi_{\alpha x}=\alpha\varphi_x$. From (3.7), applying the involution $*$ to both sides, interchanging the vectors x, y and comparing the result with (3.7) we find $\overline{\varphi_y(x)}=\varphi_x(y)$. Therefore (3.9) really defines an inner product on $\mathfrak{E}$.

Let the vector x satisfy $(x, y)=0$ for all $y \in \mathfrak{E}$. Then relations (3.9) and (3.6) yield $P_x^*=0$ i.e. $P_x=0$. Thus, in view of (3.5), $x=0$. Therefore the inner product (3.9) is non-degenerate.

Consider an operator $T \in \mathscr{A}$. From (3.5) it follows that $P_{Tx}=TP_x$ for all $x \in \mathfrak{E}$. Consequently, $P_{Tx}^*=P_x^* T^*$. Hence, making use of (3.6), we obtain $\varphi_{Tx}(y)=\varphi_x(T^* y)$ for all $y \in \mathfrak{E}$. Therefore the inner product (3.9) satisfies (2.4); in other words, it represents $\mathscr{A}$.

2. Maximal admissible *-algebras. Theorem 3.2 says that an admissible *-algebra is represented by one and, in essence, only one non-degenerate inner product. On the other hand, a non-degenerate inner product $(\cdot, \cdot)$ can represent several admissible *-algebras. It will turn out, however, that $(\cdot, \cdot)$ represents only one *maximal* admissible *-algebra.

We say the admissible *-algebras $\mathscr{A}_1, \mathscr{A}_2 \subset \mathscr{L}(\mathfrak{E})$ are *equivalent*, $\mathscr{A}_1 \sim \mathscr{A}_2$, if $\mathscr{A}_1$ and $\mathscr{A}_2$ are represented by the same non-degenerate inner products. This relation defines a partition of the class of all admissible *-algebras on $\mathfrak{E}$.

Lemma 3.3. *Let Ω be an equivalence class of admissible *-algebras on $\mathfrak{E}$. Let $(\cdot, \cdot)$ denote a non-degenerate inner product representing the elements of Ω. Then each element of Ω is a *-subalgebra of* $\mathrm{Ind}_{(\,)}$.

Proof. By definition, $(\cdot, \cdot)$ represents $\mathscr{A}$ if and only if $\mathscr{A} \subset \mathrm{Ind}_{(\,)}$ (the inclusion being meant in the sense of *-algebras).

Lemma 3.4. *If $\mathscr{A}_1, \mathscr{A}_2 \subset \mathscr{L}(\mathfrak{E})$ are admissible *-algebras such that $\mathscr{A}_1 \subset \mathscr{A}_2$, then $\mathscr{A}_1 \sim \mathscr{A}_2$.*

Proof. If $\mathscr{A}_1 \subset \mathscr{A}_2$, then the non-degenerate inner products representing $\mathscr{A}_2$ represent $\mathscr{A}_1$ too.

Theorem 3.5. *Any equivalence class Ω of admissible *-algebras on $\mathfrak{E}$ contains exactly one maximal admissible *-algebra, namely* $\mathrm{Ind}_{()}$, *where* $(\cdot, \cdot)$ *is a non-degenerate inner product representing the elements of* Ω.

Proof. Obviously, $\mathrm{Ind}_{()} \in \Omega$. Let $\mathscr{A} \subset \mathscr{L}(\mathfrak{E})$ be an admissible *-algebra with $\mathscr{A} \supset \mathrm{Ind}_{()}$. Lemma 3.4 implies that $\mathscr{A} \in \Omega$. Hence, by Lemma 3.3, $\mathscr{A} \subset \mathrm{Ind}_{()}$. Thus $\mathrm{Ind}_{()}$ is maximal.

Conversely, for any $\mathscr{A} \in \Omega$ Lemma 3.3 yields $\mathscr{A} \subset \mathrm{Ind}_{()}$. Therefore $\mathscr{A}$ cannot be maximal unless $\mathscr{A} = \mathrm{Ind}_{()}$.

From Theorems 3.2 and 3.5 we obtain:

Corollary 3.6. *The mapping* $(\cdot, \cdot) \mapsto \mathrm{Ind}_{()}$ *is a one-to-one correspondence between all non-degenerate inner products and all maximal admissible *-algebras on the same space $\mathfrak{E}$ provided we do not distinguish between inner products which are constant multiples of each other.*

Theorem 3.7. *Any admissible *-algebra $\mathscr{A} \subset \mathscr{L}(\mathfrak{E})$ can uniquely be extended to a maximal one. Namely, if $\mathscr{A}$ is represented by $(\cdot, \cdot)$, then the maximal extension is* $\mathrm{Ind}_{()}$.

Proof. According to Lemma 3.3, $\mathscr{A} \subset \mathrm{Ind}_{()}$. Theorem 3.5 assures that $\mathrm{Ind}_{()}$ is a maximal admissible *-algebra. Let $\mathscr{A} \subset \mathscr{A}_1$, where $\mathscr{A}_1$ is a maximal admissible *-algebra on $\mathfrak{E}$. Then, in view of Lemma 3.4, $\mathscr{A}_1$ is also represented by $(\cdot, \cdot)$. So, again by Theorem 3.5, $\mathscr{A}_1 = \mathrm{Ind}_{()}$.

Theorems 3.2 and 3.7 yield:

Corollary 3.8. *Two admissible *-algebras $\mathscr{A}_1, \mathscr{A}_2 \subset \mathscr{L}(\mathfrak{E})$ are represented by the same inner products if and only if $\mathscr{A}_1$ and $\mathscr{A}_2$ have the same maximal extension.*

3. Admissible *-algebras represented by quasi-definite inner products. Next we impose certain conditions on the inner product and ask the resulting features of the *-algebras they represent.

Theorem 3.9. *The non-degenerate inner products representing the admissible *-algebra $\mathscr{A} \subset \mathscr{L}(\mathfrak{E})$ are definite if and only if*

$$T^*T \neq 0 \quad (T \in \mathscr{A};\ T \neq 0). \tag{3.10}$$

Proof. Suppose the non-degenerate inner product $(\cdot, \cdot)$ represents $\mathscr{A}$, i.e.

$$(Tx, y) = (x, T^*y) \quad (T \in \mathscr{A};\ x, y \in \mathfrak{E}).$$

Let $(\cdot, \cdot)$ be definite. If for some $T_0 \in \mathscr{A}$ we have $T_0^* T_0 = 0$, then $(T_0 x, T_0 x) = (T_0^* T_0 x, x) = 0$ for all $x \in \mathfrak{E}$. Hence $T_0 = 0$.

Let $(\cdot,\cdot)$ be non-definite. Then there is a vector $z\in\mathfrak{E}$, $z\neq 0$, with $(z,z)=0$ (cf. [4; Lemma I.2.1]). As $\mathscr{A}$ is admissible, there exist an operator $R\in\mathscr{A}$ of rank 1 and an operator $Q\in\mathscr{A}$ satisfying $QR\mathfrak{E}=\langle z\rangle$. Setting $T_1=QR$ we have $(T_1^*T_1x,y)=(T_1x,T_1y)=0$ for all $x,y\in\mathfrak{E}$. Hence $T_1^*T_1=0$ though $T_1\neq 0$.

Theorem 3.10. *Let k be a non-negative integer. The non-degenerate inner products representing the admissible *-algebra $\mathscr{A}\subset\mathscr{L}(\mathfrak{E})$ are quasi-definite with rank of indefiniteness $\leqq k$ if and only if*

(3.11) $$T^*T\neq 0 \quad (T\in\mathscr{A};\ \dim T\mathfrak{E}>k).$$

Proof. Let $(\cdot,\cdot)$ represent $\mathscr{A}$.

Suppose $(\cdot,\cdot)$ is quasi-definite with rank of indefiniteness $\varkappa(\mathfrak{E})\leqq k$. If for some $T_0\in\mathscr{A}$ we have $T_0^*T_0=0$, then $(T_0x,T_0x)=(T_0^*T_0x,x)=0$ for all $x\in\mathfrak{E}$. Hence $T_0\mathfrak{E}$ is a neutral subspace, so that [2; Lemma 2] yields $\dim T_0\mathfrak{E}\leqq k$.

Suppose, conversely, that the non-degenerate inner product $(\cdot,\cdot)$ belongs to the complementary set of quasi-definite inner products with rank of indefiniteness $\leqq k$. By [2; Lemma 2] $\mathfrak{E}$ contains a neutral subspace $\mathfrak{M}$ of dimension $k+1$.

Let $x_1,\dots,x_{k+1}$ be a linearly independent system in $\mathfrak{E}$, and let $y_1,\dots,y_{k+1}$ be a basis of $\mathfrak{M}$. As $\mathscr{A}$ is admissible, it contains an operator R of rank 1. For such an R there exist vectors $x_0, y_0\neq 0$ with

$$R\mathfrak{E}=\langle y_0\rangle,\quad Rx_0=y_0.$$

Moreover, as $\mathscr{A}$ is dense, we can find operators Q_j, $S_j\in\mathscr{A}$ $(j=1,\dots,k+1)$ such that

$$Q_jx_l=\delta_{jl}x_0,\quad S_jy_0=y_j\quad (j,l=1,\dots,k+1).$$

The operator

$$T_1=\sum_{j=1}^{k+1}S_jRQ_j$$

belongs to $\mathscr{A}$ and satisfies the relations

$$T_1x_l=S_lRx_0=y_l\quad (l=1,\dots,k+1),$$

$$\dim T_1\mathfrak{E}\leqq\sum_{j=1}^{k+1}\dim S_jRQ_j\mathfrak{E}=k+1.$$

Consequently,

(3.12) $$T_1\mathfrak{E}=\mathfrak{M}.$$

As $\mathfrak{M}$ is neutral, (3.12) yields $(T_1^*T_1x,x)=(T_1x,T_1x)=0$ for all $x\in\mathfrak{E}$, and by the polarization formula (see e.g. [4; relation (I.2.3)]) also $(T_1^*T_1x,y)=0$ for all $x,y\in\mathfrak{E}$. Therefore $T_1^*T_1=0$. At the same time, $\dim T_1\mathfrak{E}>k$.

Remark 3.11. For decomposable inner products in general, the only relevant result consists of a reduction to the definite case by means of an operator $J\in\mathscr{A}$ satisfying $J^*=J^{-1}=J$ (see [3; Satz 6.1]).

§ 4. Star-algebras on Banach spaces

In this section $\mathfrak{E}$ is a Banach space over **C**. The norm of $x \in \mathfrak{E}$ will be denoted by $|x|$. Further, we denote by $\mathscr{L}(\mathfrak{E})$ the algebra of all (bounded or unbounded) linear operators on $\mathfrak{E}$, and by $\mathscr{B}(\mathfrak{E})$ the algebra of bounded linear operators on $\mathfrak{E}$.

1. Admissible *-algebras represented by continuous inner products. We are going to study how the mutual behaviour of the norm $|\cdot|$ and of a non-degenerate inner product $(\cdot, \cdot)$ is reflected on the relationship between $\mathscr{B}(\mathfrak{E})$ and $\mathrm{Ind}_{(\,)}$.

Theorem 4.1 (cf. [1; p. 196], [2; Theorem 2] and [3; Folgerung 5.6]). *The non-degenerate inner products representing the admissible *-algebra $\mathscr{A} \subset \mathscr{L}(\mathfrak{E})$ are continuous if and only if $\mathscr{A} \subset \mathscr{B}(\mathfrak{E})$.*

Proof. Suppose $(\cdot, \cdot)$ represents $\mathscr{A}$.

Let $(.,.)$ be continuous. Consider an operator $T \in \mathscr{A}$. If the sequence $(x_n) \subset \mathfrak{E}$ satisfies $x_n \to 0$ and, for some z, $Tx_n \to z$, then for all $y \in \mathfrak{E}$ we have $(Tx_n, y) \to (z, y)$ and $(Tx_n, y) = (x_n, T^* y) \to 0$; hence $z = 0$. Thus T is closed and, by the closed graph principle, bounded.

Let, conversely, $\mathscr{A} \subset \mathscr{B}(\mathfrak{E})$. According to the proof of Theorem 3.2 (see especially (3.8)) we have $(x, y) = \overline{\varphi_x(y)}\,(f, g)$, where $\varphi_x(y)$ is defined by the relation $P_x^* y = \varphi_x(y) g$ with some $g \neq 0$ and $P_x \in \mathscr{A}$. In particular, $P_x^* \in \mathscr{A}$ and, consequently, $P_x^* \in \mathscr{B}(\mathfrak{E})$. Thus $\varphi_x(y)$ and (x, y) are continuous functions of y. It follows (see subsection 2.2) that (x, y) is jointly continuous in x and y.

Setting $\mathscr{A} = \mathrm{Ind}_{(\,)}$ we find:

Corollary 4.2. *The non-degenerate inner product $(\cdot, \cdot)$ is continuous on $\mathfrak{E}$ if and only if* $\mathrm{Ind}_{(\,)} \subset \mathscr{B}(\mathfrak{E})$.

From Theorem 4.1 and Lemma 3.4 we obtain:

Corollary 4.3. *If an admissible *-algebra is contained in $\mathscr{B}(\mathfrak{E})$, then its extensions are also contained in $\mathscr{B}(\mathfrak{E})$.*

In particular:

Corollary 4.4. *If $\mathscr{B}(\mathfrak{E})$ is a *-algebra, then it is maximal.*

Theorem 4.5 (cf. [3; Satz 3.7]). *The non-degenerate inner product $(\cdot, \cdot)$ is compatible with $|\cdot|$ on $\mathfrak{E}$ if and only if* $\mathrm{Ind}_{(\,)} = \mathscr{B}(\mathfrak{E})$.

Proof. Let $(\cdot, \cdot)$ be compatible with $|\cdot|$. Consider an operator $T \in \mathscr{B}(\mathfrak{E})$. By compatibility, (Tx, y) is a continuous function of Tx and therefore, by the boundedness of T, it is a continuous function of x. Hence, again by compatibility, there exists $y_T \in \mathfrak{E}$ such that $(Tx, y) = (x, y_T)$ for all $x \in \mathfrak{E}$. Thus the adjoint $T^{(\,)}$

exists, i.e. $T \in \mathrm{Ind}_{()}$, $\mathscr{B}(\mathfrak{E}) \subset \mathrm{Ind}_{()}$. On the other hand, Corollary 4.2 yields $\mathrm{Ind}_{()} \subset \subset \mathscr{B}(\mathfrak{E})$.

Let, conversely, $\mathrm{Ind}_{()} = \mathscr{B}(\mathfrak{E})$. Then, on account of Corollary 4.2, $(\cdot, \cdot)$ is continuous. On the other hand, let φ be a continuous linear form on $\mathfrak{E}$. Set $Tx = = \varphi(x)z$ $(x \in \mathfrak{E})$, where $z \neq 0$ is fixed. Obviously, $T \in \mathscr{B}(\mathfrak{E})$. Therefore, by assumption, the adjoint $T^{()}$ exists:

$$(\varphi(x)z, y) = (x, T^{()} y)$$

for all $y \in \mathfrak{E}$. In particular, if $(z, y) = 1$, then

$$\varphi(x) = (x, T^{()} y) \quad (x \in \mathfrak{E}).$$

Theorems 4.5 and 3.7 yield:

Corollary 4.6. *The non-degenerate inner products representing the admissible *-algebra $\mathscr{A} \subset \mathscr{L}(\mathfrak{E})$ are compatible with $|\cdot|$ if and only if the maximal extension of $\mathscr{A}$ equals $\mathscr{B}(\mathfrak{E})$.*

2. Admissible *-algebras represented by decomposable, continuous inner products. In Theorems 3.9—3.10 we dealt with admissible *-algebras $\mathscr{A}$ represented by certain kinds of decomposable inner products. Below we obtain additional information in the special case $\mathscr{A} = \mathscr{B}(\mathfrak{E})$.

Our starting point is the following application of Theorem 4.5:

Theorem 4.7. *The fundamental norms corresponding to the decomposable, non-degenerate inner product $(\cdot, \cdot)$ are topologically equivalent to the given norm $|\cdot|$ on $\mathfrak{E}$ if and only if* $\mathrm{Ind}_{()} = \mathscr{B}(\mathfrak{E})$. *In this case, $\mathfrak{E}$ equipped with $(\cdot, \cdot)$ is a Krein space.*

Proof. Consider a fundamental inner product $(\cdot, \cdot)_J$ associated with $(\cdot, \cdot)$, and the corresponding fundamental norm $\|\cdot\|_J$ (see (2.6)—(2.7)). By Theorem 4.5 we must prove that $\|\cdot\|_J$ is equivalent to $|\cdot|$ if and only if $(\cdot, \cdot)$ is compatible with $|\cdot|$.

Let $\alpha_1 |x| \leqq \|x\|_J \leqq \alpha_2 |x|$ $(x \in \mathfrak{E})$, where $\alpha_1, \alpha_2 > 0$. Then $\mathfrak{E}$ is a Hilbert space relative to $(\cdot, \cdot)_J$. Since J is the difference of two orthogonal complementary projectors P^+, P^- in this Hilbert space, we have

$$|(x, y)| = |(J^2 x, y)| = |(Jx, y)_J| \leqq \|Jx\|_J \|y\|_J = \|x\|_J \|y\|_J \leqq \alpha_2^2 |x| \, |y| \quad (x, y \in \mathfrak{E}).$$

On the other hand, if the linear form φ is continuous for $|\cdot|$, then it is continuous for $\|\cdot\|_J$, so that by the Riesz representation theorem there is a $y \in \mathfrak{E}$ satisfying the relations $\varphi(x) = (x, y)_J = (x, Jy)$ $(x \in \mathfrak{E})$.

Let, conversely, $(\cdot, \cdot)$ be compatible with $|\cdot|$. Then $\|\cdot\|_J$ is continuous relative to $|\cdot|$ (see [4; Lemma IV.5.4]). Hence, if φ is a linear form continuous for $\|\cdot\|_J$, then φ is continuous for $|\cdot|$ and, consequently, there exists $y \in \mathfrak{E}$ such that $\varphi(x) = = (x, y)$ $(x \in \mathfrak{E})$. Thus $\varphi(x) = (x, J^2 y) = (x, Jy)_J$ $(x \in \mathfrak{E})$. On the other hand, by the

Schwarz inequality, $|(x, y)_J| \leqq \|x\|_J \|y\|_J$ $(x, y \in \mathfrak{E})$. As a result, $(\cdot, \cdot)_J$ is compatible with $\|\cdot\|_J$. In other words, the relation $\varphi(x) = (x, z)_J$ $(x \in \mathfrak{E})$ defines an isomorphism between $\mathfrak{E}$ and $\mathfrak{E}_J^*$, the Banach space of all linear forms which are continuous for $\|\cdot\|_J$. Moreover, by the elements of Hilbert space theory, $\|\varphi\|_J = \|z\|_J$, i.e. the isomorphism is isometrical. Therefore $\mathfrak{E}$ is complete with respect to $\|\cdot\|_J$. Once more recalling that $\|\cdot\|_J$ is continuous relative to $|\cdot|$, the closed graph principle guarantees the equivalence of $\|\cdot\|_J$ and $|\cdot|$.

From Theorem 4.7 by the aid of Theorems 3.9 and 3.10, respectively, we obtain the following results.

Corollary 4.8 (cf. [1; Corollary 4.10.8]). *The non-degenerate inner product* $(\cdot, \cdot)$ *turns* $\mathfrak{E}$ *into a Hilbert space with norm equivalent to* $|\cdot|$ *if and only if* $\mathrm{Ind}_{(\,)} = \mathscr{B}(\mathfrak{E})$ *and*

$$T^{(\,)}T \neq 0 \quad (T \in \mathscr{B}(\mathfrak{E});\ T \neq 0).$$

Corollary 4.9 (cf. [2; Theorem 3]). *Let k be a non-negative integer. The non-degenerate inner product* $(\cdot, \cdot)$ *turns* $\mathfrak{E}$ *into a Pontrjagin space with rank of indefiniteness* $\leqq k$ *and fundamental norms equivalent to* $|\cdot|$ *if and only if* $\mathrm{Ind}_{(\,)} = \mathscr{B}(\mathfrak{E})$ *and*

$$T^{(\,)}T \neq 0 \quad (T \in \mathscr{B}(\mathfrak{E});\ \dim T\mathfrak{E} > k).$$

As we have no good criterion for decomposability of inner products representing a given *-algebra (see Remark 3.11), for Krein spaces we can give only the following characterization:

Theorem 4.10. *The non-degenerate inner product* $(\cdot, \cdot)$ *turns* $\mathfrak{E}$ *into a Krein space with fundamental norms equivalent to* $|\cdot|$ *if and only if* (i) $\mathrm{Ind}_{(\,)} = \mathscr{B}(\mathfrak{E})$ *and* (ii) $\mathfrak{E}$ *is topologically isomorphic to a Hilbert space.*

Proof. Suppose that $(\cdot, \cdot)$ turns $\mathfrak{E}$ into a Krein space with fundamental norms equivalent to $|\cdot|$. Then, in particular, $\mathfrak{E}$ is decomposable, and Theorem 4.7 yields $\mathrm{Ind}_{(\,)} = \mathscr{B}(\mathfrak{E})$. Moreover, $\mathfrak{E}$ is a Hilbert space with respect to any fundamental inner product.

Suppose, conversely, that $\mathrm{Ind}_{(\,)} = \mathscr{B}(\mathfrak{E})$ and $\mathfrak{E}$ is topologically isomorphic to a Hilbert space. The norm $|\cdot|$ being involved in the theorem up to topological equivalence only, we may regard $\mathfrak{E}$ as a Hilbert space with inner product $[\cdot, \cdot]$ and norm $|x| = [x, x]^{1/2}$. On the other hand, by Theorem 4.1, $(\cdot, \cdot)$ is continuous on $\mathfrak{E}$. Consequently, there exists a bounded self-adjoint operator G on $\mathfrak{E}$ satisfying $(x, y) = [Gx, y]$ $(x, y \in \mathfrak{E})$. It is easy to see that the positive and negative spectral subspaces of G are the components of a fundamental decomposition of $\mathfrak{E}$ (cf. [4; Theorem IV.5.2]). Hence $(\cdot, \cdot)$ is decomposable and Theorem 4.7 applies.

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