Acta Sci. Math., 39 (1977), 29—30

On minimal bi-ideals and minimal quasi-ideals in compact semigroups

H. L. CHOW

In what follows, S will denote a compact topological semigroup. A non-empty subset $B\subset S$ is a *bi-ideal* of S if $BSB\cup B^2\subset B$; a non-empty subset $Q\subset S$ is a *quasi-ideal* of S if $QS\cap SQ\subset Q$. A bi-ideal (quasi-ideal) is said to be *minimal* if it does not contain properly a bi-ideal (quasi-ideal) of S. DELANGHE [1] has established the existence of minimal quasi-ideals and minimal bi-ideals in S, and, moreover, shown that the family of minimal quasi-ideals and that of minimal bi-ideals coincide. In this note, we are concerned with the relations between primitive idempotents in S and minimal quasi-ideals and minimal bi-ideals; consequently we obtain a theorem which implies all the results in [1].

An idempotent $e\in S$ is called *primitive* if e and the zero of S (which may not exist) are the only idempotents in the set eSe.

Theorem 1. *Suppose S has no zero and e is an idempotent of S. Then the following are equivalent:*

(a) *e is primitive.*

(b) *eSe is a minimal bi-ideal of S.*

(c) *eSe is a minimal quasi-ideal of S.*

Proof. (a) implies (b). Observe that eSe is a bi-ideal of S. By [3, p. 43], eSe is a group; so eSe is a minimal bi-ideal.

(b) implies (c). Let Q be a quasi-ideal of S contained in eSe. Since Q is also a bi-ideal of S, Q coincides with eSe, giving (c).

(c) implies (a). Let K be the minimal ideal of S (see [3, p. 32]); then $K\cap eSe$ is clearly a (non-empty) quasi-ideal of S and so $K\cap eSe=eSe$, whence $e\in K$. This together with [3, p. 43] completes the proof.

Received September 13, 1975.

Corollary [1]. *If B is a bi-ideal (quasi-ideal) of S, then B contains a minimal bi-ideal (minimal quasi-ideal) of S.*

Proof. We prove the corollary for bi-ideals, and the case for quasi-ideals can be derived in a similar way. First, if S has zero 0, it is obvious that $\{0\}$ is the minimal bi-ideal contained in B. Next, if S has no zero, then $B \cap K$ is a bi-ideal of S, where K denotes the minimal ideal of S. Let $x \in B \cap K$, implying that $x \in eSe$ for some idempotent $e \in K$ [3, p. 30]. Since eSe is a group containing the bi-ideal $B \cap eSe$, we see that $eSe = B \cap eSe$. Thus $eSe \subset B$, and the result follows from Theorem 1 and [3, p. 43].

Now suppose S contains a zero 0. Then an element $x \in S$ is called *nilpotent* if $x^n \to 0$ as $n \to \infty$, and a subset $A \subset S$ is said to be *nil* if every element in A is nilpotent.

Theorem 2. *If S has zero* 0 *and $e \in S$ is a non-zero idempotent of S, then the following conditions are equivalent:*

(a) *e is primitive.*

(b) *eSe is a minimal non-nil bi-ideal of S.*

(c) *eSe is a minimal non-nil quasi-ideal of S.*

Proof. The equivalence of (a) and (b) has been shown by Koch in [2], and we want to show the equivalence of (b) and (c). Suppose (b) holds, and let Q be a non-nil quasi-ideal of S contained in eSe. Since Q is also a non-nil bi-ideal, we have $Q = eSe$ so that (c) follows. Conversely, if (c) is true, then take a non-nil bi-ideal B in eSe. It is easy to see that B contains a non-zero idempotent f, in view of Lemma 2 of [2]. Hence $fSf \subset B \subset eSe$. Since fSf is a non-nil quasi-ideal of S, we have $fSf = eSe$. This yields $B = eSe$, giving the result.

References

[1] R. Delanghe, On minimal quasi-ideals and minimal bi-ideals in compact semigroups, *Acta Sci. Math.*, **36** (1974), 267—269.

[2] R. J. Koch, Remarks on primitive idempotents in compact semigroups with zero, *Proc. Amer. Math. Soc.*, **5** (1954), 828—833.

[3] A. B. Paalman-de Miranda, *Topological semigroups*, 2nd edition, Mathematisch Centrum (Amsterdam, 1970).

DEPARTMENT OF MATHEMATICS
THE CHINESE UNIVERSITY OF HONG KONG
HONG KONG