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Completeness of eigenfunctions of seminormal operators

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Let $\mathfrak{H}$ be a separable complex Hilbert space and $\mathscr{B}(\mathfrak{H})$ the algebra of bounded linear operators on $\mathfrak{H}$. An operator T in $\mathscr{B}(\mathfrak{H})$ is called a seminormal operator in case its self-commutator $D=T^*T-TT^*$ is semidefinite. In the case $D\geqq 0$ (respectively, $D\leqq 0$) the operator T is said to be hyponormal (respectively, cohyponormal). The operator T in $\mathscr{B}(\mathfrak{H})$ will be said to be completely non-normal in case the only subspace reducing the operator T on which T is a normal operator is the zero subspace. The notations $\mathrm{sp}(T)$, $\mathrm{sp}_e(T)$ and $\pi_0(T)$ will be used for the spectrum, essential spectrum and the set of eigenvalues of the operator T, respectively.

Let T be a hyponormal operator on $\mathfrak{H}$. It is easy to verify that $\ker T$ (the kernel of T) is a reducing subspace for T. Consequently, $\pi_0(T)$ must be empty whenever T is completely non-normal. On the other hand, $\pi_0(T^*)$ is sometimes non-empty. The following result will be proved in Section 1.

Theorem 1. *Let T be a completely non-normal cohyponormal operator. Assume that the planar Lebesgue measure of $\mathrm{sp}_e(T)$ is zero, then*

$$\underset{\lambda\notin \mathrm{sp}_e(T)}{\mathrm{c.l.m.}} \{\ker (T-\lambda)\} = \mathfrak{H},$$

where c.l.m.$\{\ldots\}$ *denotes the closed linear manifold generated by* $\{\ldots\}$.

If T is an operator with a rank one self-commutator, then T is either hyponormal or cohyponormal. It is still an open question as to whether such an operator T has a non-trivial invariant subspace. In certain cases T is known to possess an invariant subspace. (See [2] and [3].) On the other hand there are not many operators with a rank one self-commutator that are known to possess cyclic vectors. Theorem 1 can be used to provide examples in this direction.

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In Sections 2 and 3 we will study the singular integral operator S_b defined on $L^2(c, d)$ by

$$S_b f(s) = sf(s) + \frac{b(s)}{\pi} \int_c^d \frac{\bar{b}(t) f(t)}{t-s}\, dt,$$

where b is a non-vanishing smooth function on the interval $[c, d]$. The operator S_b is an irreducible cohyponormal operator that satisfies $S_b^* S_b - S_b S_b^* = -\frac{2}{\pi}(\ , b)b$; here, $(\ ,\ )$ denotes the inner product in $L^2(c, d)$.

In Section 3 it will be shown that b is a cyclic vector for the operator S_b. The method will entail constructing a pair of analytic continuations of the local resolvent $(S_b - \lambda)^{-1} b$ onto portions of $\pi_0(S_b)$. This leads to a discussion of solutions of singular integral equations in Section 2.

The interest in the operator S_b stems from the fact that every completely non-normal seminormal operator has a singular integral representation (see, e.g., [8], [9] and [10]).

§ 1. Completeness of eigenfunctions. PUTNAM [11] established the following remarkable inequality. Let T be a seminormal operator on $\mathfrak{H}$. Then

$$\pi \|T^* T - T T^*\| \leqq \operatorname{meas}_2(\operatorname{sp}(T)), \tag{1}$$

where meas_2 denotes planar Lebesgue measure. Below we will show how Theorem 1 follows from the inequality (1).

Proof of Theorem 1. Let $\mathfrak{M} = \underset{\lambda \notin \operatorname{sp}_e(T)}{\text{c.l.m.}} \{\ker(\lambda - T)\}$. Relative to the decomposition $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{M}^\perp$, the operator T has the matrix form

$$T = \begin{pmatrix} T_{\mathfrak{M}} & X \\ 0 & T_{\mathfrak{M}^\perp} \end{pmatrix},$$

here $T_{\mathfrak{M}}$ is the restriction of T to $\mathfrak{M}$ and $T_{\mathfrak{M}^\perp}$ denotes the compression of T to $\mathfrak{M}^\perp$. The operator $T_{\mathfrak{M}^\perp}$ is cohyponormal.

Let $\lambda \notin \operatorname{sp}_e(T)$. It follows from the continuity of the orthogonal projection onto $\ker(\mu - T)$, on the complement of $\operatorname{sp}_e(T)$, that $(\lambda - T)_{\mathfrak{M}}$ has dense range. It is easy to see that $(\lambda - T)_{\mathfrak{M}}$ has closed range and therefore $(\lambda - T)_{\mathfrak{M}}$ is onto. The surjectivity of $(\lambda - T)_{\mathfrak{M}}$ and the fact that $\ker(\lambda - T) \subset \mathfrak{M}$ imply $\lambda \notin \operatorname{sp}(T_{\mathfrak{M}^\perp})$.

The last paragraph shows that $\operatorname{meas}_2(\operatorname{sp}(T_{\mathfrak{M}^\perp})) = 0$. Thus Putnam's inequality (1) applied to the operator $T_{\mathfrak{M}^\perp}$ shows that $T_{\mathfrak{M}^\perp}$ is a normal operator. Since T is completely non-normal it must be that $\mathfrak{M}^\perp$ is the zero subspace. This completes the proof.

Let A be an operator on $\mathfrak{H}$ and let Ω be an open subset of the complex plane such that for every $\lambda \in \Omega$ the operator $A - \lambda$ is surjective. G. R. ALLAN [1] has shown that it is possible to construct an analytic right resolvent for A on Ω. This means there is a $\mathscr{B}(\mathfrak{H})$-valued analytic function $R(\lambda)$ defined on Ω such that $(A-\lambda)R(\lambda)=I$. The operator $P(\lambda)=I-R(\lambda)(A-\lambda)$ then defines an analytic projection valued function on Ω. It is clear that the range of $P(\lambda)$ is the kernel of $A-\lambda$.

Suppose now that T is an irreducible cohyponormal operator satisfying $\text{meas}_2(\text{sp}_e(T))=0$. Let $\Omega(T)=\text{sp}(T)\setminus\text{sp}_e(T)$ and assume $\Omega(T)$ is connected. Let $\{\lambda_n\}_{n=1}^{\infty}$ be an infinite sequence that accumulates in $\Omega(T)$. Then

$$\text{c.l.m.}_n \{\ker(T-\lambda_n)\} = \mathfrak{H}. \tag{2}$$

This last identity follows from Theorem 1 and the discussion in the preceding paragraph which demonstrates the existence of an analytic projection valued map onto $\ker(\lambda-T)$ for $\lambda\in\Omega(T)$.

It is interesting to note that if T is a seminormal operator and $\text{meas}_2(\text{sp}_e(T))=0$, then the self-commutator of T is compact. This follows because the projection $\hat{T}$ of T into the Calkin algebra $\mathscr{C}$ is a seminormal element in the C^*-algebra $\mathscr{C}$ with $\text{meas}_2(\text{sp}(\hat{T}))=0$. Since Putnam's inequality (1) holds for seminormal elements in any C^* algebra, then $\hat{T}$ must be a normal element in $\mathscr{C}$. This shows that T^*T-TT^* is compact. This last observation was pointed out to the author by D. D. ROGERS.

In the case where T is an irreducible cohyponormal operator with rank one self-commutator it is easy to show that the dimension of $\ker(T)$ is at most one. It follows that if X is an element commuting with T, then X leaves $\ker T$ invariant. The following is an immediate corollary of this last remark and Theorem 1.

Corollary 1. *Let T be an irreducible operator with a rank one self-commutator such that* $\text{meas}_2(\text{sp}_e(T))=0$. *Then the commutant of T is abelian.*

We remark that there are very few operators T satisfying the hypothesis of Corollary 1 for which an exact description of the commutant is known.

§ 2. Seminormal singular integral operators. Let E be a bounded measurable subset of the real line having positive measure. Let a and b be bounded measurable functions on E such that $a(t)$ is real and $b(t)\neq 0$ almost everywhere. For f in $L^2(E)$ define the singular integral operator

$$Sf(s) = sf(s)+i\left[a(s)f(s)+\frac{b(s)}{\pi i}\int_E \frac{\bar{b}(t)f(t)}{t-s}\,dt\right]. \tag{3}$$

The singular integral is interpreted as a Cauchy principal value. The operator S satisfies $S^*S-SS^*=-\frac{2}{\pi}(\,,b)b$; where $(\,,)$ denotes the inner product in $L^2(E)$.

The fact that $b(t)\neq 0$ ensures that S is irreducible. For a description of $\mathrm{sp}(S)$ and $\mathrm{sp}_e(S)$ the reader is referred to [7] and [6].

It should be remarked that if T is an irreducible cohyponormal operator with a rank one self-commutator such that the real part of T has simple spectrum, then T is unitarily equivalent to an operator of the form S. In particular, if $E=[-1, 1]$, $a\equiv 0$ and $b(t)=(1-t^2)^{1/4}$, then theoperator S defined by (3) is unitarily equivalent to the unilateral shift.

We will be concerned with the case where $E=I=[c, d]$ is an interval, $a\equiv 0$ and the function b is a non-vanishing real valued element in $C'(I)$. In this case we will denote the operator S defined by (3) as S_b. The spectrum and essential spectrum of the operator S_b can be described as follows:

$$\mathrm{sp}(S_b) = \{\lambda = \mu+i\nu : \ \mu\in I, \ |\nu| \leqq b^2(\mu)\}$$

and $\mathrm{sp}_e(S_b)$ is the boundary of $\mathrm{sp}(S_b)$. Moreover, $\pi_0(S_b)=\mathrm{sp}(S_b)\setminus\mathrm{sp}_e(S_b)$ and in view of the fact that $S_b^*S_b-S_bS_b^*$ is one dimensional, then each eigenvalue of the operator S_b has multiplicity one.

Below we will establish the existence of two analytic continuations of the local resolvent $b(\lambda)=(S-\lambda)^{-1}b\,(\lambda\notin \mathrm{sp}(S_b))$ onto portions of $\mathrm{sp}(S_b)$. In fact, we will construct two weakly analytic $L^2(I)$-valued functions b_+ and b_-, where b_+ is analytic in $J_+=(c,\infty)$ and b_- is analytic on $J_-=(-\infty, d)$, such that $(S_b-\lambda)b_\pm(\lambda)=b, \lambda\in J_\pm$. Further, $e(\lambda)=b_-(\lambda)-b_+(\lambda)$ will be a non-zero eigenfunction of the operator corresponding to λ in $J_+\cap J_-=(c, d)$.

The construction of the local resolvent necessitates solving the singular integral equation $(S_b-\lambda)x=b$. The basic method employed is discussed in the book of Tricomi [12] (see, also [4] and [5]).

Let H denote the Hilbert transform on the real line $\mathbf{R}$. Thus for $f\in L^1(\mathbf{R})$, $Hf(x)$ is defined at almost every real x by the Cauchy principal value integral

$$Hf(x) = \frac{1}{\pi}\int \frac{f(t)}{t-x}\,dt.$$

It is well known that the operator H defines a bounded linear operator on $L^p(\mathbf{R})$, $p>1$.

Let E be a bounded measurable subset of the real line and let θ be a real valued bounded measurable function supported on E. It is known that if $\exp[H\theta]$ belongs to $L^p(J)$, for some $p>1$, where J is a bounded interval containing the (essential) closure of E in its interior, then

$$\cos\theta \exp H\theta = H[\sin\theta \exp H\theta]+1. \tag{4}$$

Now for $\lambda\in J_\pm$, we define the function

$$\theta_\lambda^\pm(s) = \arg\left[\pm\frac{(\lambda-s)+ib^2(s)}{[(\lambda-s)^2+b^4(s)]^{1/2}}\right], \quad s\in I.$$

The branch of the argument is chosen such that $-\pi<\arg z\leqq\pi\,(z\neq 0)$. We remark that for λ fixed in $J_\pm$, the function $\theta_\lambda^\pm$ belongs to $C'(I)$. We will tacitly assume, whenever necessary, that the function $\theta_\lambda^\pm$ is extended to be zero off I.

Fix λ in $J_\pm$. The function $\exp H\theta_\lambda^\pm$ is easily seen to be bounded in a neighborhood of every point on $\mathbf{R}$ except possibly the points c and d. Similarly one can check that when $\lambda\in J_+$ the function $\exp H\theta_\lambda^+$ is bounded in a neighborhood of the point d and that when $\lambda\in J_-$ the function $\exp H\theta_\lambda^-$ is bounded in a neighborhood of the point c. In order to conclude that $\exp H\theta_\lambda^+$ is square integrable in a neighborhood of the point c and $\exp H\theta_\lambda^-$ is square integrable in a neighborhood of the point d, one needs only to apply Lemma 1 of [5].

Making the substitution $\theta_\lambda^\pm$ for θ in equation (4), one obtains

$$(s-\lambda)f_\lambda^\pm(s)+\frac{1}{\pi}\int_I\frac{b^2(t)f_\lambda^\pm(t)}{t-s}\,dt = 1, \quad s\in I; \tag{5}$$

here

$$f_\lambda^\pm(t) = \frac{\mp\exp[H\theta_\lambda^\pm]}{[(\lambda-t)^2+b^4(t)]^{1/2}}.$$

It follows that $b_\pm(\lambda)=bf_\lambda^\pm$ satisfies $(S_b-\lambda)b_\pm(\lambda)=b$ and further, $b_\pm(\lambda)\in L^2(I)$, for all $\lambda\in J_\pm$.

Note that for $\lambda\in J_+\cap J_-$, the function $e(\lambda)=b_-(\lambda)-b_+(\lambda)$ is a non-zero $L^2(I)$ eigenfunction of the operator S_b corresponding to the eigenvalue λ.

It is possible to extend the functions $\lambda\to\theta_\lambda^\pm$ to domains of the form

$$\hat{J}_\pm = \{\lambda=\mu+iv:\ \mu\in J_\pm,\ |v|<\varepsilon_0\},$$

where $\varepsilon_0>0$ is chosen sufficiently small. This is accomplished by defining for $\lambda\in\hat{J}_\pm$

$$\theta_\lambda^\pm(s) = \frac{1}{i}\log\left[\frac{\pm(\lambda-s)+ib^2(s)}{[(\lambda-s)^2+b^4(s)]^{1/2}}\right], \quad s\in I. \tag{5}$$

Here, if $z=re^{i\theta}, r>0, -\pi<\theta\leqq\pi$, then $\log z=\log r+i\theta$ and $\sqrt{z}=r^{1/2}e^{i\theta/2}$. The exact choice of ε_0 will depend only on the function b. The constant ε_0 is chosen such that for every s fixed on I the functions $\lambda\to\theta_\lambda^\pm(s)$ are analytic on $\hat{J}_\pm$.

It is not difficult to verify that

$$b_\pm(\lambda)(s) = \frac{\mp b(s)\exp H\theta_\lambda^\pm(s)}{[(\lambda-s)^2+b^4(s)]^{1/2}}$$

belong to $L^2(I)$ whenever $\lambda \in \hat{J}_\pm$. Moreover the map $\lambda \to b_\pm(\lambda)$ is a weakly analytic $L^2(I)$-valued mapping on $\hat{J}_\pm$. It follows that $(S_b - \lambda) b_\pm(\lambda) = b$, $\lambda \in \hat{J}_\pm$. We have therefore constructed two distinct analytic continuations b_+ and b_- of the local resolvent $(S_b - \lambda)^{-1} b$ onto $\hat{J}_+$ and $\hat{J}_-$, respectively.

§ 3. **Cyclic vectors.** We are now in a position to establish the following:

Proposition 1. *Let* $I = [c, d]$ *and assume* b *is a nonvanishing function in* $C'(I)$. *Let* S_b *be the cohyponormal operator defined on* $L^2(I)$ *by* (3). *The vector* b *is a cyclic vector for the operator* S_b.

Proof. The operator S_b is a completely non-normal cohyponormal operator with a rank one self-commutator. Moreover, $\mathrm{meas}_2(\mathrm{sp}_e(S_b)) = 0$. Let $\lambda \in (c, d)$ and let $e(\lambda)$ be the eigenfunction corresponding to the value λ described in the preceding section. It follows from the identity (2) that $\underset{c<\lambda<d}{\mathrm{c.l.m.}} \{e(\lambda)\} = L^2(I)$.

Suppose that f is in $L^2(I)$ and f is orthogonal to $\underset{n}{\mathrm{c.l.m.}} \{S_b^n b\}$, then $((S_b - \lambda)^{-1} b, f) = 0$, for $|\lambda|$ large. It follows that $(b_\pm(\lambda), f) = 0$, for $\lambda \in \hat{J}_\pm$. Consequently, $(e(\lambda), f) = 0$ for every $\lambda \in (c, d)$, and we conclude $f = 0$. This completes the proof.

It would be interesting to find the exact conditions on an element b in $C'(I)$ which ensure that b is a cyclic vector for the operator S_b. Similarly one can ask for necessary and sufficient conditions for the function b in $L^2(E)$ to be cyclic for the operator S defined by (3).

§ 4. **Conclusion.** It is not difficult to construct irreducible cohyponormal operators T such that $\mathrm{sp}(T) \setminus \mathrm{sp}_e(T)$ is non-empty and possesses the property that $\underset{\lambda \notin \mathrm{sp}_e(T)}{\mathrm{c.l.m.}} [\ker(T - \lambda)] \neq \mathfrak{H}$. The following is such an example.

Example. Let K be a perfect nowhere dense set of positive measure in [0,1], and let J be a closed interval disjoint from K. Set $E = J \cup K$ and let S_0 be the singular integral operator defined on $L^2(E)$ by (3) with the choice $a \equiv 0$ and $b \equiv 1$. The $\mathrm{sp}(S_0) = E \times [-1, 1]$ and $\mathrm{sp}(S_0) \setminus \mathrm{sp}_e(S_0)$ is the interior of $J \times [-1, 1]$. Using the usual functional calculus it is possible to obtain a non-trivial invariant subspace M for the operator S_0 such that the spectrum of S_0 restricted to M is $J \times [-1, 1]$. Any vector in $\ker(S_0 - \lambda)$ for $\lambda \in J \times I$ must be in M. It follows that $\mathrm{c.l.m.} \ker(S_0 - \lambda) \neq L^2(E)$ for $\lambda \notin \mathrm{sp}_e(S_0)$.

Corollary 1 leads to an interest in describing the commutant of an irreducible operator T with a rank one self-commutator. In particular, one can ask if the commutant of such an operator is abelian.

More specific questions can be asked about the commutants of the operators S_b, where b is a non-vanishing element in $C'(I)$. In particular, one can ask if the commutant of S_b equals the weakly closed algebra generated by S_b and the identity.

References

[1] G. R. ALLAN, Holomorphic vector valued functions on a domain of holomorphy, *J. London Math. Soc.*, **42** (1967), 509—513.

[2] C. APOSTOL and K. CLANCEY, Local resolvents of operators with one-dimensional self-commutators, *Proc. Amer. Math. Soc.* (to appear).

[3] R. W. CAREY and J. D. PINCUS, An invariant for certain operator algebras, *Proc. Nat. Acad. Sci.* (U.S.A.), **71** (1974), 1952—1956.

[4] K. CLANCEY, Completely seminormal operators with boundary eigenvalues, *Trans. Amer. Math. Soc.*, **182** (1973), 133—143.

[5] K. CLANCEY, On finite Hilbert transforms, *Trans. Amer. Math. Soc.*, **212** (1975), 347—354.

[6] K. CLANCEY, The essential spectrum of a class of singular integral operators, *Amer. J. Math.*, **96** (1974), 298—307.

[7] K. CLANCEY and C. R. PUTNAM, The spectra of hyponormal integral operators, *Comment. Math. Helvetici*, **46** (1971), 451—456.

[8] T. KATO, Smooth operators and commutators, *Studia Math.*, **31** (1968), 535—546.

[9] P. S. MUHLY, A note on commutators and singular integrals (preprint, 1975).

[10] J. D. PINCUS, Commutators and systems of singular integral equations, I. *Acta Math.*, **121** (1968), 217—244.

[11] C. R. PUTNAM, An inequality for the area of hyponormal spectra, *Math. Z.*, **28** (1970), 473—477.

[12] F. TRICOMI, *Integral Equations*, Interscience (New York, 1957).

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