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Concerning the uniqueness lemma for absolutely continuous functions

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1. We recall the classical lemma from the elements of real analysis bearing on the uniqueness of absolutely continuous functions [1], [2] which we restate in terms of vector-valued functions:

Given f: $[a, b]\to X$ *where* $-\infty<a<b<+\infty$ *and* X *is a Banach space over* R. *If* f *is absolutely continuous and* $f'(t)=0$ *(the zero element of* X*) for almost all* $t\in[a, b]$, *then* f *is constant.* (For the vector-valued situation we cannot assert in general the almost everywhere existence of $f'(t)$.)

The object of this note is to show that the lemma as stated may be established in a very simple way without the introduction of ancillary considerations such as the Vitali covering theorem or the "rising sun lemma" of F. Riesz (taken with the Hahn-Banach theorem). To be sure, these powerful approaches would appear to be indispensable to develop fully the theory of absolutely continuous functions of a single real variable and its relation to the theory of the Lebesgue integral.

2. We start with two arbitrary positive numbers ε and η in a manner reminiscent of the classical approach which uses the notion of a Vitali covering and let δ denote a positive number such that whenever $[x_k, y_k]$, $x_k<y_k$, $k=1, \dots, n$, are nonoverlapping segments in $[a, b]$ which satisfy $\sum(y_k-x_k)\leqq\delta$, we have $\sum\|f(y_k)-f(x_k)\|\leqq\eta$. Here $\|\ \|$ denotes the norm of X. Let Ω denote an open subset of R containing $[a, b]-\{f'(t)=0\}$ whose Lebesgue measure is at most δ. We introduce the class $\mathfrak{S}$, of finite sequences s that satisfy: (1) the domain of s is an initial segment $\langle 1, n(s)\rangle$ of the positive integers, (2) s maps its domain in a monotone strictly increasing fashion into $[a, b]$ with $s(1)=a$, and finally, (3) for each integer k satisfying $1\leqq k<n(s)$ either $[s(k), s(k+1)]\subset\Omega$ or

$$\|f[s(k+1)]-f[s(k)]\|\leqq\varepsilon[s(k+1)-s(k)].$$

We note that $\mathfrak{S}$ is not empty and that

$$\|f\{s[n(s)]\}-f(a)\|\leqq\eta+\varepsilon(b-a).$$

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Let $c = \sup s[n(s)]$. We are assured that

$$\|f(c) - f(a)\| \leqq \eta + \varepsilon(b-a).$$

Clearly $a < c \leqq b$ as we see on noting that either $a \in \Omega$ or $f'(a) = 0$. The assumption $c < b$ leads to a contradiction. For if $c \in \Omega$ and $s[n(s)]$ is sufficiently near c, we may extend s to a member $\sigma \in \mathfrak{S}$ with domain $\langle 1, n(s)+1 \rangle$ such that $[\sigma[n(s)], \sigma[n(s)+1]] \subset \Omega$ and $\sigma[n(s)+1] > c$, while if $c \notin \Omega$ and $s[n(s)]$ is sufficiently near c, we may this time extend s to a σ satisfying $\sigma[n(s)+1] > c$ and

$$\|f\{\sigma[n(s)+1]\} - f\{\sigma[n(s)]\}\| \leqq \varepsilon(\sigma[n(s+1)] - \sigma[n(s)]).$$

Hence $c = b$. It follows that $f(b) = f(a)$, given the arbitrariness of ε and η. The same argument applies when b is replaced by a point of (a, b).

The lemma is thereby established.

References

[1] F. Riesz—B. Sz.-Nagy, *Leçons d'analyse fonctionnelle*, Akad. Kiadó (Budapest, 1952).
[2] H. Royden, *Real analysis* (1963).

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