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An integrability theorem for power series

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1. One of the first results concerning integrability theorems for power series is due P. HEYWOOD [6] who proved that

$$\int_0^1 (1-x)^{-\gamma} f(x)\,dx < \infty \quad \text{for} \quad f(x) = \sum_{k=0}^{\infty} a_k x^k, \quad a_k \geqq 0, \quad \gamma < 1$$

if and only if

$$\sum_{n=1}^{\infty} n^{\gamma-2} \sum_{k=0}^{n} a_k < \infty.$$

A theorem, which states only an implication, was proved earlier by HARDY and LITTLEWOOD [5], as follows:

If $a_k \geqq 0$, $r \geqq p > 1$, $q > 0$ *and*

$$A(x) = \sum_{k=0}^{\infty} a_k x^k,$$

then

$$\int_0^1 (1-x)^{\frac{r}{q}-1} A^r(x)\,dx \leqq K \left(\sum_{k=1}^{\infty} k^{-\frac{p+q-pq}{q}} a_k^p \right)^{r/p},$$

where $K = K(p, q, r)$ *depends on* p, q *and* r *only.*

Henceforth — to our knowledge — P. B. KENNEDY [9], R. P. BOAS and J. M. GONZÁLEZ-FERNÁNDEZ [3], P. HEYWOOD [7], Y. M. CHEN [4], R. ASKEY [1], R. S. KHAN [10], L. LEINDLER [11], R. ASKEY and S. KARLIN [2] and P. JAIN [8] have proved similar theorems.

Very recently one of the authors ([12]) generalized most of the results known up to that time as follows:

THEOREM A. *Let* $\lambda(t) > 0$ *be a nonincreasing function on the interval* $0 < t \leqq 1$ *such that*

$$\sum_{n=k}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2} \leqq M \lambda\left(\frac{1}{k}\right) k^{-1},$$

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and let

$$F(x)=\sum_{n=0}^{\infty} c_n x^n;\quad 0\leqq x<1.$$

Suppose there is a positive monotonic sequence $\{\varrho_n\}$ *with* $\sum_{n=1}^{\infty}\frac{1}{n\varrho_n}<\infty$ *such that*

$$c_n>\frac{-K}{\left(\varrho_n\lambda\left(\frac{1}{n}\right)\right)^{1/p} n^{1-\frac{1}{p}}}\quad (0<p<\infty,\ K>0)$$

for all sufficiently large values of n. *Then* $\lambda(1-x)(|F(x)|)^p\in L(0,1)$ *if and only if*

$$\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\left(\sum_{k=0}^{n}|c_k|\right)^p<\infty.$$

In the particular case $\lambda(t)=t^{-\gamma}$ $(\gamma<1)$ and $\varrho_n=n^{\varepsilon}$ Theorem A reduces to a theorem of Jain [8], which, for $p=1$, was previously proved by Heywood [7].

In the present paper we give a generalization of Theorem A.

2. We use the following notations:

$\Phi=\Phi(p)$ $(p\geqq 1)$ denotes the set of all nonnegative functions $\varphi(u)$ having the properties: $\varphi(u)/u$ is nondecreasing and $\varphi(u)/u^p$ is nonincreasing on $(0,\infty)$.

$\Psi=\Psi(p)$ denotes the set of all functions $\psi(u)$ whose inverse functions belong to Φ.

$P=P(R)$ denotes the set of all nonnegative nondecreasing functions $\varrho(u)$ with $\varrho(u^2)\leqq R\cdot\varrho(u)$ $(u\in(0,\infty))$.

We use the notation $\bar{f}(x)$ to denote the inverse of $f(x)$.

3. We prove the following

Theorem. *Let* $\lambda(t)$ *be a positive nonincreasing function on the interval* $0<t\leqq 1$ *such that*

$$\sum_{n=k}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\leqq M\lambda\left(\frac{1}{k}\right)k^{-1} \tag{1}$$

and let $\{\alpha_n\}$ *be a positive increasing sequence with*

$$\sum_{n=1}^{\infty}\frac{1}{n\cdot\alpha_n}<\infty. \tag{2}$$

Suppose that $\varrho(u)\in P$, *that* $\eta(u)$ *denotes either a function of* Φ *or a function of* Ψ, *and that*

$$F(x)=\sum_{n=0}^{\infty}c_n x^n,\quad 0\leqq x<1. \tag{3}$$

Then, under the condition

$$c_n > -Kn^{-1}\cdot\bar{\eta}\left(\frac{n}{\alpha_n\lambda(1/n)\varrho_n}\right),\quad (K>0), \tag{4}$$

$\lambda(1-x)\eta(|F(x)|)\varrho(|F(x)|)\in L(0,1)$ *if and only if*

$$\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\varrho(n)\eta\left(\sum_{k=0}^{n}|c_k|\right)<\infty. \tag{5}$$

It is clear that this theorem includes Theorem A, namely, if $\alpha_n=\varrho_n$, $\varrho(x)\equiv 1$ and $\eta(x)=x^p$ or $\eta(x)=x^{1/p}$ $(p\geqq 1)$ then it reduces to Theorem A.

4. We require the following

Lemma. *Let* $\lambda(t)$, $\varrho(u)$ *and* $\eta(u)$ *be defined as in our Theorem, and be*

$$f(x)=\sum_{k=0}^{\infty}a_kx^k \quad \text{with} \quad a_k\geqq 0,\quad 0\leqq x<1. \tag{6}$$

Then

$$\lambda(1-x)\eta(f(x))\varrho(f(x))\in L(0,1) \tag{7}$$

if and only if

$$\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\eta(A_n)\varrho(n)<\infty; \tag{8}$$

or equivalently

$$\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\eta(A_n)\varrho(A_n)<\infty \tag{9}$$

where

$$A_n=\sum_{k=1}^{n}a_k.$$

Proof. First of all we show that (8) and (9) are equivalent.

It is easy to verify that (8) implies (9). Namely, (8) implies the existence of a natural number k such that for all $n(\geqq 2)$

$$A_n\leqq n^k$$

so the implication (8)⇒(9) is obvious. In order to show that (9) also implies (8) we use the following property of the function $\varrho(u)$ for any integer r there exists a constant C_r such that for any numbers $\alpha>0$, $\beta>0$

$$\alpha\cdot\varrho(\beta)\leqq C_r\alpha\cdot\varrho(\alpha)+\sqrt[r]{\beta}\varrho(\beta) \tag{10}$$

(this property may be proved as the statement (2.38) of Lemma 13 in [15]). Using (10) and considering that $\eta(u)/u^p\downarrow$ and $\varrho(u)\in P$ we obtain for any integer r that

$$\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\eta(A_n)\varrho(n)\leqq C_r\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\eta(A_n)\varrho(\eta(A_n))+\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}n^{1/r}\varrho(n)\leqq$$

$$\leqq K(p,r,R)\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\eta(A_n)\varrho(A_n)+\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}n^{1/r}\varrho(n). \tag{11}$$

An easy computation gives by (1) and $\varrho(u)\in P$ that

$$\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}n^{1/r}\varrho(n)<\infty \tag{12}$$

for all sufficiently large values of r. So, by (9), (11) and (12), we have (8). Now we prove the equivalence of (7) and (9). Set $y=1-x$. Since $\left(1-\frac{1}{n}\right)^n$ is an increasing sequence, we have for $\frac{1}{n+1}\leqq y\leqq\frac{1}{n}$ $(n\geqq 2)$:

$$f(1-y)\geqq\sum_{k=0}^{n}a_k(1-y)^k\geqq\sum_{k=0}^{n}a_k\left(1-\frac{1}{n}\right)^k\geqq\left(1-\frac{1}{n}\right)^n\sum_{k=0}^{n}a_k\geqq\frac{1}{4}A_n.$$

Using this we obtain for $m\geqq 2$:

$$\sum_{n=1}^{m}\lambda\left(\frac{1}{n}\right)n^{-2}\eta(A_n)\varrho(A_n)\leqq 2\sum_{n=1}^{m}\int_{\frac{1}{n+1}}^{1/n}\lambda(y)\,dy\,\eta(A_n)\varrho(A_n)\leqq$$

$$\leqq 2\int_{1/2}^{1}\lambda(y)\,dy\,\eta(A_1)\varrho(A_1)+\sum_{n=2}^{m}\int_{\frac{1}{n+1}}^{1/n}\lambda(y)\,dy\,\eta(A_n)\varrho(A_n)\leqq$$

$$\leqq O(1)+K\sum_{n=2}^{m}\int_{\frac{1}{n+1}}^{1/n}\lambda(y)\eta(f(1-y))\varrho(f(1-y))\,dy\leqq$$

$$\leqq O(1)+K\int_{0}^{1}\lambda(1-x)\eta(f(x))\varrho(f(x))\,dx.$$

This proves that (9) follows from (7). To prove the inverse statement of the equivalence we consider the following estimations for $m \geqq 1$

$$\int_0^{1-\frac{1}{m+1}} \lambda(1-x)\eta(f(x))\varrho(f(x))\,dx = \sum_{n=1}^{m} \int_{\frac{1}{n+1}}^{1/n} \lambda(y)\eta(f(1-y))\varrho(f(1-y))\,dy =$$

$$= \sum_{n=1}^{m} \int_{\frac{1}{n+1}}^{1/n} \lambda(y)\eta\left(\sum_{k=0}^{\infty} a_k(1-y)^k\right)\varrho\left(\sum_{k=0}^{\infty} a_k(1-y)^k\right)dy \leqq$$

(13)
$$\leqq \sum_{n=1}^{m} \int_{\frac{1}{n+1}}^{1/n} \lambda(y)\eta\left(\sum_{k=0}^{\infty} a_k\left(1-\frac{1}{n+1}\right)^k\right)\varrho\left(\sum_{k=0}^{\infty} a_k\left(1-\frac{1}{n+1}\right)^k\right)dy \leqq$$

$$\leqq O(1)\sum_{n=1}^{m} \lambda\left(\frac{1}{n}\right) n^{-2}\eta\left(\sum_{k=0}^{\infty} a_k\left(1-\frac{1}{n+1}\right)^k\right)\varrho\left(\sum_{k=0}^{\infty} a_k\left(1-\frac{1}{n+1}\right)^k\right).$$

Since $\frac{1}{2} \geqq \left(1-\frac{1}{n+1}\right)^n \geqq \left(1-\frac{1}{n+2}\right)^{n+1}$ for $n=1, 2, \ldots$ we have

$$\sum_{k=0}^{\infty} a_k\left(1-\frac{1}{n+1}\right)^k \leqq \sum_{j=0}^{\infty}\sum_{k=nj}^{n(j+1)} a_k\left(1-\frac{1}{n+1}\right)^k \leqq$$

(14)
$$\leqq \sum_{j=0}^{\infty}\left(1-\frac{1}{n+1}\right)^{nj}\sum_{k=nj}^{n(j+1)} a_k \leqq 2\sum_{i=1}^{\infty} 2^{-i} A_{ni}.$$

Henceforth we split the proof into two parts. If $\eta(u)=\varphi(u)$ then we use the inequality

(15)
$$\varphi\left(\frac{\sum_{i=1}^{\infty} a_i b_i}{\sum_{i=1}^{\infty} a_i}\right)\varrho\left(\frac{\sum_{i=1}^{\infty} a_i b_i}{\sum_{i=1}^{\infty} a_i}\right) \leqq K\cdot\frac{\sum_{i=1}^{\infty} a_i\varphi(b_i)\varrho(b_i)}{\sum_{i=1}^{\infty} a_i}.$$

This property of the function $\varphi(u)\varrho(u)$ immediately follows from results of H. P. Mulholland [13] (see Theorem 1 and Remark (2.34)) and from the properties of the functions $\varphi(u)$ and $\varrho(u)$. By (15) we get:

(16)
$$\varphi\left(\sum_{i=1}^{\infty} 2^{-i}A_{ni}\right)\varrho\left(\sum_{i=1}^{\infty} 2^{-i}A_{ni}\right) \leqq K\sum_{i=1}^{\infty} 2^{-i}\varphi(A_{ni})\varrho(A_{ni}).$$

Hence and from (13), (14) and (16) we deduce, for $m\geqq 1$, that

$$\int_0^{1-\frac{1}{m+1}} \lambda(1-x)\varphi(f(x))\varrho(f(x))\,dx \leqq O(1)\sum_{n=1}^{m}\lambda\left(\frac{1}{n}\right)n^{-2}\sum_{i=1}^{\infty}2^{-i}\varphi(A_{ni})\varrho(A_{ni}) \leqq$$

$$\leqq O(1)\sum_{i=1}^{\infty}2^{-i}\sum_{n=1}^{m}\lambda\left(\frac{1}{n}\right)n^{-2}\varphi(A_{ni})\varrho(A_{ni}) \leqq$$

$$\leqq O(1)\sum_{i=1}^{\infty}2^{-i}i^2\sum_{n=1}^{m}\lambda\left(\frac{1}{ni}\right)(ni)^{-2}\varphi(A_{ni})\varrho(A_{ni}) \leqq O(1)\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\varphi(A_n)\varrho(A_n).$$

If $\eta(u)=\psi(u)$ the proof runs similarly but we use the following inequality

$$\psi\left(\sum_{i=1}^{\infty}2^{-i}A_{ni}\right)\varrho\left(\sum_{i=1}^{\infty}2^{-i}A_{ni}\right) \leqq O(1)\sum_{i=1}^{\infty}2^{-\frac{i}{p}}\psi(A_{ni})\varrho\left(\sum_{i=1}^{\infty}\frac{(ni)^t}{2^i}\right) \leqq \tag{17}$$

$$\leqq O(1)\sum_{i=1}^{\infty}2^{-\frac{i}{p}}\psi(A_{ni})\varrho(n)$$

instead of (16). Inequality (17) is just an easy consequence of the following elementary facts:

$$\psi(a+b)\leqq\psi(a)+\psi(b),\quad \psi(kx)\leqq k^{1/p}\psi(x)\quad\text{for}\quad k<1,$$

and that, by (8), there exists an integer t such that $A_n\leqq n^t$ for any $n(\geqq 2)$. Thus the proof of Lemma is completed.

5. Proof of the theorem.

Let $A(x)=\sum_{k=0}^{\infty}a_kx^k$ for $0\leqq x<1$ with $a_0=0$ and

$$a_k = K\cdot k^{-1}\cdot\bar{\eta}\left(\frac{k}{\alpha_k\lambda(1/k)\varrho(k)}\right).$$

First we consider the case $\eta(u)=\varphi(u)$.

We show that these coefficients a_k satisfy condition (8). Using the inequality

$$\sum_{n=1}^{\infty}\lambda_n\varphi(A_n)\leqq K_1\sum_{n=1}^{\infty}\lambda_n\varphi\left(\frac{a_n}{\lambda_n}\sum_{k=n}^{\infty}\lambda_k\right) \tag{18}$$

which holds for any $\lambda_n>0$ and $a_n\geqq 0$ (see the inequality (8) of [14]) with $\lambda_n=\lambda(1/n)n^{-2}\varrho(n)$, and the following consequence of (1)

$$\sum_{n=k}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\varrho(n)\leqq M\lambda\left(\frac{1}{k}\right)k^{-1}\varrho(k)$$

we have

$$\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\varphi(A_n)\varrho(n)\leqq O(1)\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\varrho(n)\varphi(n\cdot a_n)\leqq$$

$$\leqq O(1)\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\varrho(n)n\frac{1}{\alpha_n\lambda(1/n)\varrho(n)}\leqq O(1)\sum_{n=1}^{\infty}\frac{1}{n\cdot\alpha_n}<\infty.$$

Hereby we proved that the coefficients of the function $A(x)$ satisfy condition (8), so by Lemma

(19) $$\lambda(1-x)\varphi(A(x))\varrho(A(x))\in L(0,1).$$

By (4) the coefficients a_n+c_n are positive for all sufficiently large values of n, thus the functions

$$A(x)+F(x)=\sum_{n=0}^{\infty}(a_n+c_n)x^n$$

has the property

(20) $$\lambda(1-x)\varphi(A(x)+F(x))\varrho(A(x)+F(x))\in L(0,1)$$

if and only if

(21) $$\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\varphi\left(\sum_{k=0}^{n}(a_k+c_n)\right)\varrho(n)<\infty.$$

If $\lambda(1-x)\varphi(|F(x)|)\varrho(|F(x)|)\in L(0,1)$, then (19) implies (20) which implies (21). But by (4) we have

$$|c_n|\leqq 2a_n+c_n,$$

whence, by (8) and (21), (5) follows.

If (5) holds, then this implies (21) because from (15) immediately follows that

(22) $$\varphi(a+b)\varrho(a+b)\leqq K(\varphi(a)\varrho(a)+\varphi(b)\varrho(b)),\quad a>0,\ b>0.$$

But from (21) follows (20). By (19) and (20)

$$\lambda(1-x)\varphi(|F(x)|)\varrho(|F(x)|)\in L(0,1)$$

follows obviously.

Thus the theorem is proved for $\eta(u)=\varphi(u)$. The proof for $\eta(u)=\psi(u)$ runs similarly. To prove (8) we use the inequality

$$\psi(a+b)\leqq\psi(a)+\psi(b)\quad\text{for all}\quad a>0,\ b>0;$$

thus

$$\sum_{n=2}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\varrho(n)\psi\left(\sum_{k=1}^{n}a_k\right)\leqq\sum_{m=0}^{\infty}\sum_{n=2^m+1}^{2^{m+1}}\lambda\left(\frac{1}{n}\right)n^{-2}\varrho(n)\psi\left(\sum_{k=1}^{2^{m+1}}a_k\right)\leqq$$

$$\leqq O(1)\sum_{m=0}^{\infty}\lambda\left(\frac{1}{2^{m+1}}\right)2^{-m}\varrho(2^{m+1})\psi\left(\sum_{k=1}^{m+1}\bar{\psi}\left(\frac{2^k}{\lambda(1/2^k)\alpha_{2^k}\varrho(2^k)}\right)\right)\leqq$$

$$\leqq O(1)\sum_{k=1}^{\infty}\frac{2^k}{\lambda(1/2^k)\alpha_{2^k}\varrho(2^k)}\sum_{m=k}^{\infty}\lambda\left(\frac{1}{2^{m+1}}\right)2^{-m}\varrho(2^m)\leqq O(1)\sum_{k=1}^{\infty}\frac{1}{\alpha_{2^k}}<\infty.$$

From this point the proof runs on the same line as before. The proof is thus completed.

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